Gravitational Instability

\[ \sigma_2(r) = \frac{a \Omega^2(r)}{2G}, \]  

where \( \sigma_2 = 5.35 \text{ kpc}, \, V_0 = 413 \text{ km/s} \)

According to the dynamical theory of stable stellar systems \([6]\), the dispersion velocity of the stars in the radial direction \( \sigma_2 \) is a constant; for example, we can take \( \sigma_2 = 45 \text{ km/s} \)

Also, the theory of an axisymmetric, self-gravitating, turbulent gaseous disk \([7]\) says that the turbulent velocity of the gas is proportional to the rotational velocity,

\[ \sigma_2 \propto n \nu(r), \]

in which \( n \) is the constant of proportionality. Since the rotational velocity of the Milky Way, \( \nu = 250 \text{ km/s} \), can be taken as approximate constant over a considerable range of \( r \), if we take \( n = 0.04 \), then we have

\[ \sigma_2 \leq 10 \text{ km/s} \]

From these we find

\[ \Omega_2 - \frac{K_2}{G \sigma_2} = \frac{2\sigma_2}{V_0} (4 + r^2/d^2)(1 + r^2/d^2), \]

\[ \Omega_1 - \frac{K_1}{G \sigma_2} = \frac{2\sigma_2}{V_0} (4 + r^2/d^2)(1 + r^2/d^2). \]

In the paper by Lin et al. \([8]\), after fixing \( \Omega(r) \) and \( \sigma_2 \), an independent assumption was made regarding the value of \( \sigma_2 \), and this is, in general, not self-consistent. For, if we have the rotation curve \( \Omega(r) \), then we can solve the zero-order equation of motion and the Poisson's equation for the surface density distribution \( \frac{\rho_2}{\sigma_2^2} [\tilde{r}] \), \([9]\), and hence calculate \( \rho_2 \).

The gas content of the Milky Way system at 10 kpc from the centre is about 20%. From (35) and (36), we see that \( \sigma_2 \approx 0.9 \) and \( \rho_2 \approx 0.205 \), both less than 1, hence obviously unstable. Fig. 1 shows that, if the dispersion velocity of the gas does not vary, then the minimum dispersion velocity of the stars required for stability should be raised several fold. However, at \( r = 5 \text{ kpc} \), we have \( \sigma_2 \approx 1.4 \), and \( \rho_2 \approx 0.31 \), and if the fraction of gas \( 

As we have not allowed for the effect of the finite thickness of the system, our calculated values of the minimum dispersion velocity are on the large side; allowing for the thickness would greatly complicate the calculations and its effect is, we estimate, only to modify somewhat the shape of the limiting stability curves on the \( \Omega_2 = \Omega_1 \)-diagram.

REFERENCES

[4] Ibid., p115
2. Properties of Polytopes

1. A Generalized Lane-Emden Equation

The basic equations we consider are the hydrostatic equation and Poisson's Equation:

$$\rho \frac{\partial v}{\partial t} + \nabla p = 0,$$
$$\Delta \phi = 4\pi G \rho,$$

in which $G$ and $\phi$ are the gravitational constant and potential, $P$ and $\rho$ are the pressure and density. We assume $P$ and $\rho$ to satisfy the polytropic equation of state

$$p = K \rho^{1 + n},$$

in which $n$ is the polytropic index and $K$ is a constant.

We shall replace $n$ and $K$ by $\alpha$ and $H$, defined by

$$\alpha = \left(1 + n\right)^{-1}, \quad H = \frac{K}{4\pi G}.$$

The equation of state is now written

$$p = H \rho^{\alpha}.$$

Inserting (5) into (1), and integrating, we have

$$\Phi - \Phi_0 = -4\pi G \rho^{\alpha} \left(F - F_0\right).$$

In (6), $\Phi_0$ and $F_0$ are the potential and pressure at the reference point. Eliminating $\rho$, we have

$$H \rho^{\alpha} - \frac{1}{\alpha} \frac{\partial}{\partial r} \left(1 - \frac{p}{p_0^{\alpha}}\right) = \frac{p}{p_0^{\alpha}}.$$  \hspace{1cm} (7)

Introducing the non-dimensional quantities $\xi$ and $\phi$:

$$\xi = \frac{r}{r_d}, \quad \phi = \frac{1 - (p/p_0)^{\alpha}}{\alpha},$$

$\rho$ being the spatial coordinate with the dimension of a length, and $r_d = \left(\frac{H}{4\pi G}\right)^{1/\alpha} p_0^{1/\alpha}$ being the scale factor. In terms of these non-dimensional quantities, (7) is written as

$$\Delta \phi = (1 - \alpha \phi) \frac{\partial \xi}{\partial \xi},$$

$$\Delta \phi = \left(1 - \frac{\phi}{n + 1}\right)^{n + 1}.$$  \hspace{1cm} (9)

Equation (9) is the general form of Emden's equation. This form is applicable to any real value of $n$ (apart from -1) including $z = \alpha$ (the isothermal case), for $n = m$, (9) becomes

$$\xi = \left(1 + \frac{1}{2\beta n}\right)^{n+1} - \frac{\phi}{n+1}. \hspace{1cm} (10)$$

equation (9) can be changed into an alternative form

$$\Delta \phi = \left(\frac{1}{\alpha (n+1)} \frac{\partial}{\partial \xi}, \right.$$

which is not as general as the form (9) in the variables $(\xi, \phi)$. However, we shall have occasions to use the variables $(z, \xi)$. Considering that we shall be applying (9) to the three symmetrical geometrical shapes.

write (9) in the form

$$\phi'' + \frac{n - 1}{n + 1} \phi' = \left(1 - \frac{\phi}{n + 1}\right)^{n + 1}.$$  \hspace{1cm} (11)

The gravitational potential energy $U_\rho$ and the internal energy $U_\rho$ of the mass $M_\rho$ can be found from the definitions to be

$$S_\rho = \frac{2 \Gamma (1/2)}{\Gamma (m/2)} \frac{2}{2\pi n} = \frac{2}{2\pi n}.$$  \hspace{1cm} (14)
Polytropes

\[ B, m = \int_{r_{1}}^{r} \frac{\partial \Phi}{\partial r} dM, \]
\[ = \Phi r dM \left\{ \left( \frac{\xi^{2}}{\mu^{2}} \right) \xi_{0}^{2} - (2 - m) \int_{r_{1}}^{r} \xi^{2} \mu^{2} d\xi \right\} \]
\[ U, = -c_s^2 \int_{r_{1}}^{r} T dM, \]
\[ = \frac{B_0}{T - 1} \int_{r_{1}}^{r} \xi^{2} \mu^{2} d\xi \]
\[ \text{where } T \text{ is the temperature, } \sigma_{v} \text{ is the specific heat at constant pressure, and } \gamma \text{ is the ratio of specific heats. For the integrals on the right side of (15), an integration by parts gives} \]
\[ \int_{r_{1}}^{r} \xi^{2} \mu^{2} d\xi = \frac{n + 1}{2m - (m - 2)(n + 1)} \left\{ 2m \left[ \xi^{2} \xi_{0}^{2} \right] \xi_{0}^{2} + 2 \left[ \xi^{2} \xi_{0}^{2} \right] \xi_{0} \right\} \]
\[ + \frac{1}{2m - (m - 2)(n + 1)} \left\{ -(n + 1)(m - 2) \left[ \xi^{2} \xi_{0}^{2} \right] \xi_{0} \right\} \]
\[ + 2 \left[ \xi^{2} \xi_{0}^{2} \right] \xi_{0} + \left[ \xi^{2} \xi_{0}^{2} \right] \xi_{0} \right\} \]
\[ \text{For the temperature, we assume the intermediate medium to be an ideal gas, } P_{\text{gas}} = \frac{\mu}{\nu} R T, \]
\[ F \text{ gas being the gas pressure, } R \text{ the gas constant, and } \nu \text{ the mean molecular weight. If the only pressure is that due to the gas, then} \]
\[ \frac{T}{\mu} = \left( \frac{\nu}{\mu} \right) \frac{\partial \Phi}{\partial \nu}, \text{ with } \frac{T}{\nu} = \frac{H}{R} \frac{p_{\text{gas}}}{\nu}. \]

3. Homology Invariants \( \nu \) and Geometrical Properties of the \( \nu \)-plane

Since (11) admits homology transformations, we define two homology invariants
\[ u = \frac{dn M}{dn r}, \quad v = \frac{\partial \Phi}{\partial \nu}. \]
\[ \text{Differentiating logarithmically with respect to } \xi, \]
\[ \frac{du}{dn \xi} = m - u + \left( \frac{n}{n + 1} \right) v \equiv \Phi(u, v), \]
\[ \frac{dv}{dn \xi} = 2 - m + u + \left( \frac{1}{n + 1} \right) v \equiv \Phi(u, v), \]
\[ \text{and writing } \xi = \ln \xi \text{ in (10), we have} \]
\[ \frac{dx}{dv} = u \Phi(u, v) = D(u, v), \]
\[ \frac{dx}{ds} = v \Phi(u, v) = N(u, v), \]
\[ \text{The equations (19) are of the same form as the equations of motion for a point } \nu_{\text{m}}, \text{ on the } \nu \text{-plane, with the independent variable } \xi \text{ corresponding to the time variable, and } (du/ds, dv/ds) \text{ to the velocity vector of } \nu_{\text{m}}. \]

In these differential equations, the independent \( \xi \) is implicit; eliminating \( \xi \) from them, we obtain the integrated curve on the \( \nu \)-plane:
\[ \frac{dv}{du} = \frac{N(u, v)}{D(u, v)} \]
\[ \text{However, equation (20) will not fix the direction of motion, so if we wish to know the oriented trajectory of } \nu_{\text{m}}, \text{ then we must resort to (19). According to the conditions for a singular point, } \nu = 0, \text{ the singular points of (20) are the intersections of two sets of lines with gradients } 0 \text{ and } m, \text{ namely, } A(0, 0) X(n, 0) \text{ and } B(0, n - 2) X(1, n - 1). \]

Linear differential equations have, in general, four types of singularities. The equations (20) are not linear, but the type of singularity can be investigated by developing around the singular point. Let \( (u', v') \) represent the coordinates of a singular point, take the translations \( u = u' + \alpha u, \quad v = v' + \beta v, \) where \( u, v \) are small quantities; since the most important terms in determining the properties of a singular point are the linear terms, we have
\[ \frac{dv}{du} = \frac{b u + b v}{a u + v}, \]
\[ \text{where} \]
\[ a_s = m - 2u, \quad b_s = -\frac{n}{n + 1}, \quad a_v = -\frac{n}{n + 1}, \]
\[ b_v = v. \]

Let \( \lambda_1 + \lambda_2 = 0, \quad q_1 = q_2 = 0, \quad \text{the type of singularity is then determined by the nature of the roots} \]
\[ (\lambda_1, \lambda_2) = \frac{1}{2} [q_1 \pm (q_1^2 + 4q_2)^{1/2}], \]
\[ \text{If } \lambda_1 \text{ and } \lambda_2 \text{ are real and of the same sign, then the singularity is a node, if they are real and have opposite signs, then it is a saddle point; if the roots are pure imaginaries, then the singularity is a vortex; if they are complex conjugates, then it is a focus (see Fig. 1). For given values of } \mu, \nu, \text{ the types to which the various singularities belong can be seen at a glance from Fig. 1.}^* \]

Apart from those 4 singular points, the other possible end-points for the trajectories are the 4 points at infinity in the physically meaningful first quadrant: \( B(0, -\infty), \quad C(0, +\infty), \quad D(\infty, +\infty), \quad \text{and the extremities corresponding to the } n = 0 \text{ locus, } D(n, +\infty). \]

*Translator's Note: A translation of the expression for \( q_2 \) given in the original text has been reversed in order to be consistent with Fig. 1. A corresponding reversal is made in (22). The sense of \( q_1 \) in Fig. 1 has also been reversed.
Polytropes

Fig. 1 Types of singular points. A, X, Y, G mark the singular points. Suffix indicates shape, 1 for slab, 2 for cylinder, 3 for sphere.

Table 1 Properties of End-Points

<table>
<thead>
<tr>
<th></th>
<th>$n$, $v_1$</th>
<th>$n$, $v_1$</th>
<th>$t$, $n$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$0$, $0$</td>
<td>$m$, $v_2$</td>
<td>$n$, $x+y$</td>
<td>$t$, $n$</td>
</tr>
<tr>
<td>$X$</td>
<td>$n$, $0$</td>
<td>$v_1$, $v_2$</td>
<td>$x+y$</td>
<td>$t$, $n$</td>
</tr>
<tr>
<td>$Y$</td>
<td>$(m-1)(m+2)$</td>
<td>$(m-1)(m+2)$</td>
<td>$(m-1)(m+2)$</td>
<td>$t$, $n$</td>
</tr>
<tr>
<td>$G$</td>
<td>$m$, $v_1$</td>
<td>$m$, $v_2$</td>
<td>$x+y$</td>
<td>$t$, $n$</td>
</tr>
</tbody>
</table>

Fig. 2 The Imden solutions on the $a$, $m$-plane

--- is the locus of the singular point $G$.
--- is the locus of the critical point of gravitational instability.

Classifying the integrated curves according as their initial points are $B$, $X$, $A$, $Y$, or $G$, we obtain 5 types of solutions (TABLE 2).

Starting from one of these points, the curve is integrated according to given values of $m$ and $n$ to reach $A$, $B$, $X$, $Y$, or $G$, the shape of the curve being determined by the type of singularity of the singular point for the given values of $m$ and $n$. Figs. 2 and 5 show the families of Imden solutions and the "hollow-centre" type solutions.

Analytical and asymptotic solutions of the generalised Lane-Emden Equation (11) are given in the Appendix.
### Table 2: Solutions of the Basic Equation (11)

<table>
<thead>
<tr>
<th>$P_i$</th>
<th>$I$</th>
<th>$J$</th>
<th>$C$</th>
<th>$Y$</th>
<th>$G$</th>
<th>Type of Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>$\sigma &gt; 1$</td>
<td>$\sigma = 1$</td>
<td>$0 &lt; \phi &lt; 1$</td>
<td>$\sigma_1 = 0$</td>
<td>$\sigma_2 = 0$</td>
<td>$\frac{\alpha - 1}{\beta} &gt; 0$</td>
</tr>
<tr>
<td>$X$</td>
<td>$\sigma &gt; 1$</td>
<td>$\sigma = 1$</td>
<td>$0 &lt; \phi_1 &lt; 1$</td>
<td>$\sigma_1 = 0$</td>
<td>$\sigma_1 = 0$</td>
<td>$\frac{\alpha - 1}{\beta} &gt; 0$</td>
</tr>
<tr>
<td>$A$</td>
<td>$\sigma &gt; 1$</td>
<td>$\sigma = 1$</td>
<td>$0 &lt; \phi_1 &lt; 1$</td>
<td>$\sigma_1 = 0$</td>
<td>$\sigma_1 = 0$</td>
<td>$\frac{\alpha - 1}{\beta} &gt; 0$</td>
</tr>
<tr>
<td>$Y$</td>
<td>$\sigma &gt; 1$</td>
<td>$\sigma = 1$</td>
<td>$0 &lt; \phi_1 &lt; 1$</td>
<td>$\sigma_1 = 0$</td>
<td>$\sigma_1 = 0$</td>
<td>$\frac{\alpha - 1}{\beta} &gt; 0$</td>
</tr>
</tbody>
</table>

$P_i$ are initial points. $P_f$ are final points. $A, B, Y, G$ are singular points. $S_1, S_3, C$ are points at infinity in the first quadrant. "\(\pi\)" marks the closed curve around $G$.

<table>
<thead>
<tr>
<th>$P_i$</th>
<th>$I$</th>
<th>$J$</th>
<th>$C$</th>
<th>$Y$</th>
<th>$G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d \ln P_i}{d \ln \xi_1}$</td>
<td>$\frac{d \ln P_i}{d \ln \xi_1}$</td>
<td>$\frac{d \ln P_i}{d \ln \xi_1}$</td>
<td>$\frac{d \ln P_i}{d \ln \xi_1}$</td>
<td>$\frac{d \ln P_i}{d \ln \xi_1}$</td>
<td>$\frac{d \ln P_i}{d \ln \xi_1}$</td>
</tr>
<tr>
<td>$\frac{d \ln M_i}{d \ln \xi_1}$</td>
<td>$\frac{d \ln M_i}{d \ln \xi_1}$</td>
<td>$\frac{d \ln M_i}{d \ln \xi_1}$</td>
<td>$\frac{d \ln M_i}{d \ln \xi_1}$</td>
<td>$\frac{d \ln M_i}{d \ln \xi_1}$</td>
<td>$\frac{d \ln M_i}{d \ln \xi_1}$</td>
</tr>
<tr>
<td>$\frac{d \ln r_i}{d \ln \xi_1}$</td>
<td>$\frac{d \ln r_i}{d \ln \xi_1}$</td>
<td>$\frac{d \ln r_i}{d \ln \xi_1}$</td>
<td>$\frac{d \ln r_i}{d \ln \xi_1}$</td>
<td>$\frac{d \ln r_i}{d \ln \xi_1}$</td>
<td>$\frac{d \ln r_i}{d \ln \xi_1}$</td>
</tr>
</tbody>
</table>

From (17) defining $u$ and $v$, we have

- $u_1 = (m - 1) \frac{d \ln q_i}{d \ln \xi_1}$
- $v_1 = - (n + 1) \frac{d \ln q_i}{d \ln \xi_1}$

Substituting into (26), and simplifying, we have

- $Z = \frac{d \ln P_i}{d \ln \xi_1}$
- $Z = \frac{d \ln M_i}{d \ln \xi_1}$
- $Z = \frac{d \ln r_i}{d \ln \xi_1}$

Since we are considering disturbances at constant mass, $Z = u_1 (\psi_2 / \psi_0)$; inserting this in (28), we have

- $\frac{d \ln P_i}{d \ln \xi_1} = \frac{d \ln M_i}{d \ln \xi_1} = \frac{d \ln r_i}{d \ln \xi_1}$

Eliminating $\xi$ from (29), we have

- $f_0' (u_1, v_1) = f_0'' (u_1, v_1)$

The condition for instability (25) thus becomes

- $f_0'' (u_1, v_1) > 0$
The expression (31) shows that the singular point \( S \) is a key point and that the region of instability, in virtue of this expression, can now be treated as a geometrical problem in the \( \Sigma \)-plane. The region of instability lies between the two straight lines \( \beta_{p} = 0 \) and \( \beta_{*} = 0 \), where \( f'_{\beta} > 0 \) (shown shaded in Fig. 4). The criterion (31) is applicable to all the solutions of TABLE 2.

(a) \( \sigma > 1 \) (\( \rho > 1 \))
(b) \( 1 > \sigma > \frac{1}{2} \) (\( \beta < \beta_{c} \))
(c) \( \beta > \frac{1}{2} \) (\( \beta > \beta_{c} \))
(d) \( \beta < \frac{1}{2} \) (\( \beta < \beta_{c} \))

Fig. 4 The region of instability (31) is shown shaded. \( E \) marks the Emden solution (cf. Fig. 2).

2. Stability of Polytropes of the Emden Solutions

Referring to Fig. 2, we see that the Emden solutions can be simply divided into 3 cases:
(a) When \( \sigma > 1 \) (\( \rho > 1 \)), the entire \( \Sigma \)-locus lies in the unstable region. This range of the index corresponds to the condition of thermal instability pointed out by Field [4] \( \frac{4 \ln T}{\ln \rho} = \frac{1}{n} < -1 \), hence it is a region of thermal instability (Fig. 4a).
(b) When \( \sigma = 1 \) (\( n = 0 \)), the matter is incompressible, and no contraction can take place.
(c) When \( 1 > \sigma > 1/2 \) (\( 1 > n > 0 \)), we have \( \beta_{p} < \beta_{c} \), the \( \Sigma \)-locus can appear only in the stable region (Fig. 4c).
(d) When \( 1 > \sigma > \frac{1}{2} \) (\( 1 > n > 0 \)), \( \beta_{c} \) is situated in the second quadrant, and the \( \Sigma \)-locus lies entirely in the stable region (Fig. 4d).
(e) When \( \sigma < \frac{1}{2} \) (\( n < 1 \)), the \( \Sigma \)-locus is in the stable or unstable region according to whether the mass \( M \) is being less or greater than a critical mass \( M_{c} \). Here the stability character depends mainly on the mass, so that it is a question of gravitational instability (Fig. 4e).

(a) illustrates the polar coordinates \( (X, R) \) used, with \( X = r^{n-1} \), \( R = r^{n-1} \), \( n = 1 \) \( R = m \leq m_{*} \).
(b) \( E \) marks the region of unconditional stability, and \( G \) the region where it is stable or unstable according to the mass being less or greater than the critical mass. \( \beta \) is the region of thermal instability.

Fig. 5 Stable and unstable regions

In the figure, \( M \) marks the critical mass of a complete polytrope, \( M_{c} \), that of an incomplete polytrope (both normed). The values of \( M_{c} \) correspond to those marked in Fig. 2 by the dot-dash lines.

Fig. 6 Dependence of the critical mass on \( n \) and \( m \).
In view of the above results on stability, we introduce the parameter $X_n$:

$$X_n = \frac{a + 1}{n} \quad \text{or} \quad \sin X_n = \frac{a + 1}{a},$$

in terms of which we can state the dependence on $n$ of the $\dot{P}/P$ diagram customary used in evolution theories, for $\chi_n$ is the angle of the inclination on the $\dot{P}/P$ diagram. The whole range of variation of $n (-\infty < n < +\infty)$ corresponds to a semi-circle $X_n \leq \chi_n \leq \chi_n$ (S) and the three geometrical shapes are mapped on to the semi-circle of Fig. 5a.

This semi-circle is divided into three parts (Fig. 2b): if the indices $m, n$ of an incomplete polytrope fall in Region S, then it is stable; if in Region G, then thermal instability will occur; if in Region D, then it depends on whether its mass is greater or less than a critical mass for a gravitational instability to arise.

In the end, we note that the singular point $\dot{\eta}$, the curve passes through $\dot{X}$, the critical mass $H_{\dot{\eta}}$ that corresponds to the smallest value of $\dot{X}$, which we denote as $H_{\dot{\eta}}$. We then show the results of our numerical solution in Fig. 6, in which $H_{\dot{\eta}}$ represents the critical mass for a complete polytrope.

The above discussion was on incomplete polytropes. For complete polytropes, we can also use the semi-circle of Fig. 5a to make a classification according to the finiteness or otherwise of the radius and mass. The result is Fig. 7, with details given in the appended table.

![Fig. 7 Diagram for complete polytropes](image)

- $r_1$, both the radius and mass are finite;
- $r_1$, radius infinite, mass finite; in $r_1$, both are infinite. Cf. Fig. 5a for reading of $m$, $n$ values.

4. CRITERION OF JEANS' INSTABILITY

The Jeans' criterion is a criterion for instability against small density disturbances in an infinite, homogeneous medium, assuming isothermal changes. An infinite homogeneous medium may be stable for all values of the polytropic index including -1. In this paper, we generalise Jeans' criterion to cases where the changes are not isothermal.

$$c\varphi^2 \sim 4\pi G\rho_n < 0 \quad (32)$$

where $\varphi$ is the sound speed, $c\varphi^2 = \left(\frac{dP}{d\rho}\right)_T$, and $k$ is the wave number, and the suffix $l$ refers to values in the equilibrium state. The sound speed $c_1$ is related to the polytropic index by

$$c_1 = \left(\frac{dP}{d\rho}\right)_T = \frac{a + 1}{n} \frac{P_n}{\rho_n}, \quad (33)$$

The critical wavelength is the one that corresponds to the equal sign in (32), that is

$$R_c = 2\pi \frac{a + 1}{n} \frac{P_n}{\rho_n} \quad (34)$$

The so-called Jeans' wavelength is obtained when we let $n = +\infty$ in (34):

$$R_N = 2\pi a \frac{a + 1}{n} \quad (35)$$

and the ratio of the two is

$$R_N/R_j = \left(\frac{a + 1}{n}\right)^{1/3} \quad (36)$$

5. CONCLUSIONS

In this paper, after defining two variables $\varphi = (n-1)(1-\varphi)$ and $\varphi = 1/n^2 \varphi$ to replace the usual polytropic function $\varphi$ and the variable $\varphi$, we have obtained the general form of the Lane-Emden function of index $n$, which is applicable to all real values of $n$ including $\varphi = 1$ (the isothermal case), and excluding $-1$ (the isobaric case). We have taken three symmetrical shapes for the polytropes, the slab, the cylinder and the sphere. We have analysed the properties of the end-points (including the singular points), and have given in Table 1 the different types of solutions classified according to their starting points. Those starting from $X(n,p)$, and with $m = 3$, and $n > 0$ (the end solutions) have already proved to be of great value in the study of the internal structure and evolution of stars. For the study of nebulae, the cases with $n < 0$ may be of importance, for, according to the cloud phase, the heat source comes from outside. We have called the solutions starting from the point $S = 0$, "Hollow centre" solutions which correspond to the initial conditions $\varphi = 1$, $\varphi = 1 + \varphi$ at $\zeta = 0$, that is, a model of a heavenly body with a relatively low central density. The "loaded" solutions in Table 2 were proposed by Huntley and Saslaw [5] as models for galactic nuclei with massive cores.

On the question of stability of polytropes, we have considered the radial disturbances in incomplete polytropes and obtained the expression (31) expressing the condition of instability. The advantage of this criterion is that it has converted the question of stability into one of geometry in the $\omega$-plane. For the end solutions, using the parameter $X_m = \frac{a + 1}{n}(n+1)/n$, we have carried out on the complete real axis of the polytropic index (except the point -1) and for these symmetrical shapes ($n=1,2,3$), a stable region $S$, a gravitational unstable region $G$ and a thermally unstable region $S$ (Fig. 2b). Gravitational instability is so-called because it is determined by a critical mass, while thermal instability is determined by the condition of thermal instability. We have also analysed the question of finiteness of radius and mass of complete polytropes, and the results were shown in Fig. 6. We have also discussed the dependence of the critical mass on $n$ and the results shown in Fig. 6.
1. Analytical Solutions

Analytical solutions of equation (11) exist in the following cases:

(a) When \( n = 0 \) or 1, (11) is a linear equation,

\[
\theta_0 = a + b \left[ \sum_{k=0}^{m} \frac{z^k}{2^{m-k}} \right], \quad \theta_1 = \sin \left( \frac{\zeta}{\sqrt{2}} \right),
\]

\[
\psi_0 = \frac{1}{\sqrt{2}} \left[ a \cos \left( \frac{\zeta}{\sqrt{2}} \right) - b \sin \left( \frac{\zeta}{\sqrt{2}} \right) \right], \quad \psi_1 = \frac{1}{\sqrt{2}} \left[ a \sin \left( \frac{\zeta}{\sqrt{2}} \right) + b \cos \left( \frac{\zeta}{\sqrt{2}} \right) \right],
\]

(b) In the endom solutions, the integration constants are \( a = 1 \), and \( b = 0 \).

(c) For \( n = 1 \), (11) can be solved by separation of variables.

\[
\zeta = \frac{x}{\sqrt{2}} \int \frac{1}{\sqrt{1 + \beta \nu^2}} d\nu,
\]

Endom solution takes \( a = 1 \) and \( \zeta = 0 \). The integral can be analytically evaluated when \( 2n(n+1) \) is an integer; when \( n + 1 > 0 \), it is an incomplete \( \beta \) function; in other cases, it can only be evaluated numerically.

(d) When \( \frac{1}{n+1} = \frac{m-2}{2m} \), we can find an integrating factor \( \nu^m \), \( \alpha = m/2 - 1 \), \( \alpha + n/2 \), making (20) a total differential equation, which is then integrated to give

\[
\left( \nu \right)^m \nu^2 = \left( \frac{2n}{n+1} \right)^{1/2},
\]

For the endom solution,

\[
\psi = \frac{1}{\sqrt{2}} \left[ a \sin \left( \frac{\zeta}{\sqrt{2}} \right) + b \cos \left( \frac{\zeta}{\sqrt{2}} \right) \right],
\]

The \((\nu, \psi)\) locus at the point \( C \) is a singular solution. To be physically meaningful, \( n \) must have \( -1 < \frac{n}{\alpha-1} < \frac{m-2}{m} \), and

\[
\psi = \frac{1}{\sqrt{2}} \left[ a \sin \left( \frac{\zeta}{\sqrt{2}} \right) + b \cos \left( \frac{\zeta}{\sqrt{2}} \right) \right].
\]