THE WHITAKER-VONDRAK METHOD OF DATA SMOOTHING AS A NUMERICAL FILTER

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ABSTRACT We investigate the properties of the Whitaker-Vondrak method of data smoothing as a numerical filter. We obtain the theoretical expression relating the smoothing parameter $c$, the period $P$ and the frequency response $A$, thus verifying the result found by Vondrak, and generalize the expression to other definitions of "smoothness". Using the noise spectra before and after the filter, we give the formulae for estimating the mean error per unit weight, the mean error of a filtered value, and the autocorrelation in the filtered series.

INTRODUCTION

The method of smoothing observational data series proposed by Whitaker and Robinson [1] on the basis of probability theory, after being further improved and developed by Vondrak [2, 3], is now widely applied in the treatment of astronomical data, especially in the field of orbital service work. We have used this method in the preliminary reduction of doppler observations of artificial satellites with good results. Vondrak found that, when applied to astronomical time series, this method resembled a one-sided frequency filter. In this paper we attempt to clarify the theoretical properties of the method as a numerical filter.

While the theory and practice of the least-squares errors are well understood, opinions differ on filtering problems when the data is smoothed by a filter if the errors are not equal to each other, we can regard the filtered value as dependent on the underlying errors. Also, Vondrak proposed that, if the mean square per unit weight $n$ is known beforehand, then the best value of $c$ is that for which $\sum_{i=1}^{n} r_i^2 / n = n$. But the difficulty is that $n$ is often not known in advance, but has to be found from the smoothed values. These are the questions we shall be dealing with.

THE WHITAKER-VONDRAK METHOD

In forming in using a mathematical function to approximate a time series we must do not know which function would be most appropriate. We must regard the problem as a question of probability, namely, how to find the most probable curve from the observed series.

In the Whitaker-Vondrak method, the most probable curve is taken to be some best compromise between the "degree of fidelity" and "smoothness". Mathematically, the curve is to be found from the requirement

\[ F = F_c + c = \min \]

\[ F = \frac{1}{2} \sum (y_i - \hat{y}_i)^2 \]

\[ S = \int (\gamma - \gamma_i)^2 \frac{1}{y_i} dy_i \]

and the "degree of fidelity" and the "degree of smoothness", defined in terms of the smoothed values $y_i$, the observed values $y_i$, the weights $w_i$ and the number of observed values. To the third-order in the parameter $c$ we have

\[ y(x) = y_0 + \frac{1}{2} \gamma_0 x + \frac{1}{6} \gamma_2 x^2 + \frac{1}{24} \gamma_4 x^4 + \ldots \]

\[ (\gamma_{20} = \gamma_2 x_0 + \gamma_4 x_0^2 + \gamma_6 x_0^3 + \ldots) \]

\[ \gamma_0 = \frac{1}{2} \sum y_i, \gamma_2 = \frac{1}{2} \sum (y_i - \gamma_0)^2, \gamma_4 = \frac{1}{24} \sum (y_i - \gamma_0)^4, \ldots \]

\[ \gamma_{20} = \gamma_2 x_0 + \gamma_4 x_0^2 + \gamma_6 x_0^3 + \ldots \]
Inserting (4) in (3) gives

$$S = (1 - r)^{-1} \sum_{j=1}^{n} \left( a_j + b_j \frac{y_j}{E_j} + c_j \frac{y_j}{E_j} \right)^2,$$  \[ (5) \]

where

$$a_j = \frac{6 \sqrt{E_{j1} - E_{j2}}}{(E_{j1} - E_{j2}) (x_j - x_{j2}) (x_j - x_{j3})},$$

$$b_j = \frac{6 \sqrt{E_{j2} - E_{j3}}}{(E_{j2} - E_{j3}) (x_j - x_{j1}) (x_j - x_{j3})},$$

$$\ldots$$

Putting (2) and (5) in (1), we then have the equations for determining the smoothed values $y$, $y'$, $y''$, etc.

$$\frac{\partial S}{\partial y_i} = 0, \quad i = 1, 2, \ldots, k.$$  \[ (6) \]

The matrix of coefficients being almost diagonal, the solution is easily effected. If $k^2 = m$, then the solution is simply $y = y'$, i.e., absolutely faithful; if $k^2 = m$, then the solution is a parabola and $S = 0$, i.e., absolutely smooth. Between these extremes, we have some eclectic solution. Vondrak found that for cosine functions of various periods $p$, the frequency response $F(x, p)$ of $e^{-i x / p}$ is the same for $p^2 \equiv \text{const.}$, and is in the form of a low-pass filter. We shall therefore call this the "Whittaker-Vondrak" filter and proceed to examine its properties.

3. THE WHITTAKER-VONDRAK FILTER

Since the smoothed values are linear combinations of the observed values, the filter is a linear filter. Hence if the observed series can be expressed as the sum of a smooth function and a number of periodic terms, we can consider each periodic term in isolation. Consider, then, an input sine curve with unit amplitude and period $p$,

$$y = \sin \frac{2\pi}{p} (x + \phi),$$  \[ (7) \]

Assuming $p \ll T$ (T being the total length of the sample) and assuming sufficiently dense sampling, it can be proved using fundamentals that the solution of (1), i.e., the output curve, will also be a sine curve with amplitude $A$ and zero phase-shift,

$$y = A \sin \frac{2\pi}{p} (x + \phi)$$  \[ (8) \]

Hence the degree of fidelity is

$$F = \frac{\int_{0}^{T} \left( y - y' \right)^2 \, dt}{\int_{0}^{T} \left( y - y' \right)^2 \, dt} = 1 - \frac{A^2}{\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2}}$$  \[ (9) \]

Here, $A$ corresponds to the frequency response $F(x, p)$ of Ref. 31, $A = 1$ means total pass, $A = 0$ means total rejection.

Consider now the degree of smoothness of $y$. We measure the independent variable in units of 2, and assume the sampling to be at equal intervals of $h = 1/2m$, i.e., at the points

$$x = 0, \quad \frac{1}{2m}, \quad \frac{3}{2m}, \quad \ldots, \quad \frac{2m - 1}{2m}.$$  \[ (10) \]

From (4), we then have

$$y''(x) = \frac{1}{h^2} \left( -y_{n+1} + 2y_{n+1} - 3y_n + 2y_{n+1} - y_{n+1} \right)$$

$$= \frac{1}{h^2} \left( \sin \frac{2\pi}{p} \left( \frac{1}{2m} + \phi \right) - \sin \frac{2\pi}{p} \left( \frac{1}{2m} + \phi \right) - 2 \sin \frac{2\pi}{p} \left( \frac{1}{2m} + \phi \right) + 2 \sin \frac{2\pi}{p} \left( \frac{1}{2m} + \phi \right) \right),$$

or, using well-known trigonometric formulas,

$$y''(x) = -\frac{96}{p^2} \cos \frac{2\pi}{p} \left( \frac{1}{2m} + \phi \right) \left( \sin \frac{2\pi}{p} \frac{1}{2m} \right).$$  \[ (11) \]

Substituting (11) in (3) gives

$$S = \frac{32 \pi^2}{p^4} \sin \frac{2\pi}{p} \left( \frac{1}{2m} + \phi \right) = 32 \pi^2 \frac{\alpha}{p^4} \sin \frac{2\pi}{p} \left( \frac{1}{2m} + \phi \right).$$  \[ (12) \]

This shows that the higher the frequency, the rougher will be the output, the degree of smoothness being proportional to the 6th power of the frequency. With (9) and (11), condition (1) becomes

$$A = \frac{1}{2} \left( 1 - \frac{A^2}{\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2}} \right) = 0, \quad \text{min.}$$  \[ (13) \]

This to hold, we require

$$A = \frac{1}{1 + 64 \pi^2 \sin^2 \frac{2\pi}{p} \frac{1}{2m}} = \frac{1}{1 + 64 \pi^2 \sin^2 \frac{2\pi}{p} \frac{1}{2m}} \quad (14)$$

In numerical form of this relation is given in Table 1. This result is basically the same as that found empirically in Ref. 31, thus we have proved that the Whittaker-Vondrak method is a low-pass filter. It is easily seen that this result depends simply on the property of the filter.

<table>
<thead>
<tr>
<th>$k$</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
<th>1.0</th>
<th>10</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin^2 \frac{2\pi}{p} \frac{1}{2m}$</td>
<td>0.01</td>
<td>0.1</td>
<td>1.0</td>
<td>10</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

The table and will hold whether the sampling is at equal or unequal intervals. If we change the definition of smoothness, using the $k$-th order difference instead of the third order difference, then we shall get

$$S = \frac{1}{4} A \left( \frac{2\pi}{p} \frac{1}{2m} \right)^2 = \frac{1}{4} A \left( \frac{2\pi}{p} \frac{1}{2m} \right)^2.$$  \[ (15) \]

By using the sensitive dependence of $S$ on $A$, inserting this in (1) gives

$$A = \frac{1}{1 + 64 \pi^2 \sin^2 \frac{2\pi}{p} \frac{1}{2m}} \frac{1}{1 + 64 \pi^2 \sin^2 \frac{2\pi}{p} \frac{1}{2m}} \quad (16)$$

The different frequency response curves for selected values of $k$ and one value of $k = \infty$ are shown in Fig. 1. We note that the larger $k$ is, the steeper $A$ rises and the more nearly ideal the filter will be, though of course, the computation becomes more complicated at the same time.

4. ESTIMATING THE ERRORS IN THE OUTPUT VALUES OF A NUMERICAL FILTER

Numerical smoothing of observed data is essentially applying a low-pass filter to filter out the high-frequency components regarded as errors in the time series. The choice of the cutoff frequency depends on the physical properties of the data. Since, in general, observational error has the characteristic of white noise, a higher cutoff frequency will mean less error being filtered out, smaller residuals, and more error remaining in the filtered series. Therefore, under the premise of not throwing away any useful information, we should use a cut-off frequency that is as low as possible. Based on this line of thought, we have established the relations between the mean error of unit weight and residuals and between the mean error of a filtered value and residuals.
The Whittaker-Vondrak Method

First, consider the frequency characteristics of a discrete sampling noise. Let the sample size be \( N = 2n + 1 \), the sampling interval be \( h = 1/2m \), and the sampling be at the points \( x_j = jh \). The Fourier expansion of the noise \( v_j \) is

\[
V_j = \frac{2}{2N} \sum_{k=1}^{N} \left( a_k \cos(kx_j) + b_k \sin(kx_j) \right) \tag{17}
\]

\[
a_k = \frac{2}{2m + 1} \sum_{j=1}^{N} V_{kj} \cos(kx_j) \tag{18}
\]

\[
b_k = \frac{2}{2m + 1} \sum_{j=1}^{N} V_{kj} \sin(kx_j) \tag{19}
\]

Let \( V_j \) be a white noise (\( E(V) = 0, \) \( \text{Var}(V) = \sigma^2 \)). Expression (18) shows that \( a_k/2 \) is simply the sample mean. In satellite work, \( N \) is so large that the central limit theorem certainly holds, hence \( a_k/2 \) follows a normal distribution with

\[
E\left( \frac{a_k}{2} \right) = 0, \quad D\left( \frac{a_k}{2} \right) = \frac{\sigma^2}{2m + 1} = \frac{\sigma^2}{N}. \tag{20}
\]

This means, when smoothing data with a filter, even if we filter out the fluctuations at all frequencies, we shall still have not eliminated all errors -- \( \sigma a_k/2 \) may be said to represent the upper bound of the accuracy of any numerical filter, i.e.,

\[
\sigma a_k = \frac{1}{\sqrt{N}} \sigma_v. \tag{22}
\]

Expressions (19) and (20) show that \( a_k \) and \( b_k \) are linear functions of the random variable \( v_j \), so that they will be Gaussian variables if the latter is. The expectation operator and the linear operator being commutative, we have

\[
E(a_k) = \frac{2}{2m + 1} \sum_{j=1}^{N} \cos(kx_j) E(V_j) = 0, \tag{23}
\]

\[
E(b_k) = \frac{2}{2m + 1} \sum_{j=1}^{N} \sin(kx_j) E(V_j) = 0. \tag{24}
\]

Since the sampled values are mutually independent, so are \( a_n \) and \( b_m \), and moreover,

\[
\sigma_a^2 = \frac{2}{2m + 1} \sum_{j=1}^{N} \cos(kx_j) \sigma_v^2, \tag{25}
\]

\[
\sigma_b^2 = \frac{2}{2m + 1} \sum_{j=1}^{N} \sin(kx_j) \sigma_v^2, \tag{26}
\]

hence the variance at frequency \( n \) is

\[
\sigma_n^2 = \frac{2}{N} \sigma_v^2 \left( \cos(2\pi nx_j) + \sin(2\pi nx_j) \right) = \frac{2}{N} \sigma_v^2, \tag{27}
\]

Combining (21) and (25), we find the power spectrum in a discrete sampling of white noise random variables to be

\[
\frac{1}{N} \sigma_v^2, \quad \frac{2}{N} \sigma_v^2, \quad \frac{2}{N} \sigma_v^2, \ldots, \quad \frac{2}{N} \sigma_v^2
\]

We suppose the filter to be an ideal low-pass filter, i.e., with gain \( A \) of the form

\[
A = \begin{cases} 1, & 0 < h, \\ 0, & h > f_s. \end{cases} \tag{28}
\]

Here \( f_s \) is the cutoff frequency. Let \( k \) be the largest integer equal to or less than \( f_s \). The remaining noise variance after filtering, that is, the variance in the smoothed value is

\[
\sigma_f^2 = \sigma_v^2 \left( \frac{2}{N} \right) \tag{29}
\]

and the variance filtered out is

\[
\sigma_f^2 = \frac{2m + 1 - (1 + 2k)}{N} \sigma_v^2, \tag{30}
\]

\[
\sigma_f^2 = \frac{2m + 1 - (1 + 2k)}{N} \sigma_v^2. \tag{31}
\]

\[
\sigma_f^2 = \frac{2m + 1 - (1 + 2k)}{N} \sigma_v^2. \tag{32}
\]

\[
\sigma_f^2 = \frac{2m + 1 - (1 + 2k)}{N} \sigma_v^2. \tag{33}
\]

\[
\sigma_f^2 = \frac{2m + 1 - (1 + 2k)}{N} \sigma_v^2. \tag{34}
\]

\[
\sigma_f^2 = \frac{2m + 1 - (1 + 2k)}{N} \sigma_v^2. \tag{35}
\]

\[
\sigma_f^2 = \frac{2m + 1 - (1 + 2k)}{N} \sigma_v^2. \tag{36}
\]

In the last expression, the form of a damped oscillation. Similarly, after passing through the Whittaker-Vondrak filter, the autocorrelation function is
The Whittaker-Vondrak Method

\[ R(x) = \frac{1}{2\pi} \int \frac{1}{(1 + \xi^2)} e^{-i\xi x} \, d\xi \]

where \( \xi \) represents the power spectrum of the original white noise. A rigorous integration is cumbersome, but for illustrating the general characteristics, we can replace it by an ideal filter with cutoff frequency \( f_0 \). Table 2 gives the result for \( f_0 = 1/60 \) sec per a.

<table>
<thead>
<tr>
<th>( x(x) )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<th>16</th>
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<th>19</th>
<th>20</th>
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<tr>
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<td>23</td>
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<td>( \nu(x) )</td>
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6. CONCLUSIONS

1. The Whittaker-Vondrak method of smoothing observational series is essentially a one-sided numerical filter, its frequency response is given at (16).
2. The characteristics of the filter are closely related to the definition of 'smoothness' and different filters can be constructed to suit the particular physical properties of the observed series.
3. The expressions we derived for the mean error of unit width and the mean error of a filtered variable are, in principle, applicable to any numerical filter.

REFERENCES


CAN THE EQUAL-ALTITUDE METHOD MAKE NEW CONTRIBUTIONS?

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ABSTRACT. This paper advances the thesis that, by using the double angle-mirror system, by choosing the instrumental almanac to pass through the celestial pole, and by simultaneous recording of the time and the azimuth of the star being observed, we can greatly increase the effectiveness of equal-altitude instruments in star catalogue work.

INTRODUCTION

The plan of space astrometry has aroused great interest, by using a purpose-built astrometric satellite such as Hipparcos, precision in the determination of star positions, proper motions, and parallaxes is expected to increase by orders of magnitude. However, since many factors, e.g., there will be a transition period between ground-based astrometry and space astrometry. During this period, it is a necessary and urgent task to develop suitable ground-based instruments of high precision and performance that can provide an initial base.

In 1981, Guinot et al. [2] used a new astrolabe observations published a well-known catalog of 571 stars. It had a precision comparable to that of the FK5 catalogue and limited strong responses from the international community. In the last ten or more years, there has been new development in the equal-altitude method and instrument and we feel it is ripe for renewed discussion.

PROBLEMS PRESENT IN DETERMINING STAR COORDINATES USING THE EQUAL-ALTITUDE METHOD

The basic equation in making an astrologist star catalogue is

\[ \xi = \tan \alpha = \tan \delta \]

where \( \xi \), \( \alpha \), and \( \delta \) are the mean star-group parameters representing corrections required to place the star in the catalog. \( \alpha \) and \( \delta \) are corrections to the ideal system \( \alpha_0 \) and \( \delta_0 \) to the ideal system where \( \xi_0 \) is the mean residual of the program star relative to the mean star system. For a "twice-crossover" closing the almanac once on the \( \xi_0 \) gives

\[ \Delta \alpha = \frac{\Delta \alpha_0 - \Delta \alpha_0}{\cos q} + \Delta \alpha_0 \]

(2)

\[ \Delta \delta = \frac{\Delta \delta_0 + \Delta \delta_0}{\cos q} + \Delta \delta_0 \]

(3)

where \( \Delta \alpha_0 \) is the right ascension zero, which can only be determined from observations of \( \alpha \) in the solar system, as in the meridian method \( \Delta \xi = 0 \), is a constant to be determined by the method. When \( \Delta \xi \) is less than 0.2, twice crossovers are not available in short observing runs.

From (2) and (3), the problems present can be seen to be the following:

1. The presence of a systematic error in the form \( \Delta \alpha_0 \) in the determination of \( \alpha_0 \). Here, the functional form is known, but the coefficient is unknown. It differs from the usual errors in transit instruments in that the latter has a more complicated form.
2. The accuracy of the determination of \( \Delta \alpha_0 \) and \( \Delta \delta_0 \). It is serious in the declination than in the right ascension. A sizeable blind area is often present in the latest-determined positions.
3. Only observations of twice-crossovers can isolate the two components and these are not available in short observing runs.

In 1968, Kleinlein [3] pointed out that, by comparing the latitude observations obtained by the method at the same location working with different almanacs, it is possible to find systematic errors in the declinations. We have commented on the accuracy attainable by the method in the determination of the constant \( \Delta \xi_0 \).

DEVELOPING THE PRINCIPLES OF THE EQUAL-ALTITUDE METHOD

We solve the above problems in the determination of star coordinates, and to increase the