MONOCHROMATIC IMAGES IN STOKES PARAMETERS AND
THE STRUCTURE OF MAGNETIC FIELDS IN SUNSPOTS

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Abstract. Taking into account magneto-optical effects, we have obtained numerical solutions of the transfer equations for the Stokes parameters, calculated the linearly polarized intensity ($\bar{U}$) and constructed its monochromatic images of unipolar sunspots. By comparison with the observational material of the vector magnetograph of the Marshall Space Flight Center, Huntsville (Alabama), we have found that the model of radial magnetic fields may give rise to $\bar{U}$ monochromatic images close to those observed. The same conclusion has been obtained previously by Landi Degl'Innocenti (1979), although his analysis was performed with the Milne–Eddington approximation instead of a detailed sunspot model. Moreover, we have shown that the model of spiral magnetic fields leads to results in contrast with observations.

1. Introduction

Various models of sunspot magnetic fields have been established in the past, and the one most commonly used is the fan-shaped model (e.g., Hale and Nicholson, 1938). According to this model the magnetic lines of force exhibit a radial distribution. Yet some observers, such as Ding You-ji et al. (1976), have found that white-light images of sunspots sometimes possess spiral structures. This may possibly mean that magnetic fields in spots have a spiral structure. Considering the difficulties of measuring the directions of magnetic fields in sunspots, it is desirable to find an indirect method of ascertaining the configuration of these sunspot magnetic fields. In recent years the vector magnetograph of the Marshall Space Flight Center (Hagyard and Cumings, 1975) has measured linearly polarized intensity distributions of unipolar sunspots in the $\bar{Q}$ and $\bar{U}$ parameters* of the magneto-sensitive spectral line Fe i $\lambda$5250.216. (In this paper we call such intensity distributions the monochromatic images in Stokes parameters.) It is striking to see that these images exhibit conspicuous spiral structures. As the two Stokes parameters $Q$ and $U$ represent linear polarization, and as the direction of polarization is closely connected with the direction of the magnetic field, Hagyard et al. (1977) quite naturally thought that sunspot magnetic fields were spiral in structure. However, Landi Degl'Innocenti (1979) pointed out that if magneto-optical effects are taken into account, even a purely radial magnetic field model may give rise to spiral $\bar{Q}$ and $\bar{U}$ monochromatic images. But he did not explain the configuration that will be given by a spiral model nor how it differs from the images yielded by a radial model. Besides, in the computation of $\bar{Q}$ and $\bar{U}$ he used the Milne–Eddington atmospheric model and a linear distribution of the Planck function with optical depth. All these are rather crude approximations. Later, West and Hagyard (1983) also considered magneto-optical effects, but they used

* The exact definitions of these quantities are given later by formulae (5).

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Unno's algebraic solution of the transfer equations for the Stokes parameters, with some improvements. As we have shown (Ye Shi-hui et al. 1978), the algebraic solution is less accurate than our numerical solution. Besides, their comparison with observations is qualitative in character, i.e., only the geometrical shape of the regions with positive or negative $Q$ (or $U$) is investigated, while the absolute values of these two quantities are not considered. In our opinion, all these problems should be studied to a greater extent by means of a more appropriate sunspot model and a more accurate solution of the transfer equations.

2. Magneto-Optical Effects and Signal of Linear Polarization

Under the action of magneto-optical effects, the Stokes parameters $I$, $Q$, $U$, $V$ are determined by the following Unno–Beckers equations:

\[
\begin{align*}
\cos \theta \frac{dl}{dr} &= (1 + \eta_t) (I - B) + \eta_Q Q + \eta_U U + \eta_V V, \\
\cos \theta \frac{dQ}{dr} &= \eta_Q (I - B) + (1 + \eta_t) Q - \rho_R U + \rho_W \sin 2\chi V, \\
\cos \theta \frac{dU}{dr} &= \eta_U (I - B) + \rho_R Q + (1 + \eta_t) U - \rho_W \cos 2\chi V, \\
\cos \theta \frac{dV}{dr} &= \eta_V (I - B) - \rho_W \sin 2\chi Q + \rho_W \cos 2\chi U + (1 + \eta_t) V.
\end{align*}
\]

The meanings of the various symbols may be found in the works of Unno (1956) and Beckers (1969). It has been recognized by some authors (e.g., Landi Degl’Innocenti (1976)) that there are some errors in the original form of the equations given by Beckers (1969), and we have made the right corrections. For the central part of the solar disk we may take $\cos \theta = 1$. In the above system of equations the parameters which represent magneto-optical effects are

\[
\begin{align*}
\rho_R &= \frac{\eta_0}{H(0, a)} \cos \gamma [F(v - v_H, a) - F(v + v_H, a)], \\
\rho_W &= \frac{\eta_0}{H(0, a)} \frac{\sin^2 \gamma}{2} \left[ F(v - v_H, a) + F(v + v_H, a) - 2F(v, a) \right],
\end{align*}
\]

where

\[
F(v, a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{u}{u^2 + a^2} e^{-(u - v)^2} du.
\]

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Moreover,

\[ \eta_I = \frac{\eta_p}{2} \sin^2 \gamma + \frac{\eta_b + \eta_r}{4} (1 + \cos^2 \gamma), \]

\[ \eta_Q = \left( \frac{\eta_p}{2} - \frac{\eta_b + \eta_r}{4} \right) \sin^2 \gamma \sin 2\chi, \]

\[ \eta_U = \left( \frac{\eta_p}{2} - \frac{\eta_b + \eta_r}{4} \right) \sin^2 \gamma \sin 2\chi, \]

\[ \eta_V = \frac{\eta_b - \eta_r}{2} \cos \gamma. \]

(4)

In order to find the emerging Stokes parameters, we used the sunspot atmospheric model given by Kneer (1972), adopted a series of values for each of the three parameters of the magnetic vector – i.e. the strength \( B \), the angle between its direction and the line of sight \( \gamma \) and the azimuthal angle \( \chi \) – and obtained numerical solutions of the Unno–Beckers equations for the magneto-sensitive lines FeI \( \lambda \lambda 6302.499 \) and 5324.191 (Ye Shi-hui et al., 1983). According to our calculations, magneto-optical effects cannot be neglected for sunspots. Now we proceed to compute the signals of linear polarization \( \tilde{Q} \) and \( \tilde{U} \) for the vector magnetograph of MSFC. By the definitions in Hagyard et al. (1977) these signals are given by the following expressions:

\[ \tilde{Q} = \int Q(\lambda)f(\lambda) \, d\lambda / \int I(\lambda)f(\lambda) \, d\lambda, \]

\[ \tilde{U} = \int U(\lambda)f(\lambda) \, d\lambda / \int I(\lambda)f(\lambda) \, d\lambda, \]

where \( f(\lambda) \) is the transparency profile of the narrow-band filter, i.e.

\[ f(\lambda) = C \exp \left[ - \left( \frac{\lambda - \lambda_0}{\Delta \lambda_F} \right)^2 \right], \]

(6)

where

\[ \Delta \lambda_F = 125/2 \sqrt{\ln 2} \approx 75 \text{ mÅ}. \]

(7)

By use of these formulae, as well as Kneer’s atmospheric model, one may calculate theoretical monochromatic \( \tilde{Q} \) and \( \tilde{U} \) images for various models of sunspot magnetic fields.

3. Models of Sunspot Magnetic Fields

As we have said in the Section 1, the most frequently used model is the fan-shaped model, which can be represented by the following formulae:

\[ B(\rho) = \frac{B_0}{1 + \rho^2}, \]

(8)
\[ \gamma(\rho) = \frac{\pi}{2} \rho, \]  

where \( B_0 \) is the magnetic field strength at spot center, \( \rho \equiv r/R \), \( R \) is the radius of the circular spot, and \( r \) is the distance of an arbitrary point in the spot from the center.

Hagyard et al. (1977) established a spiral model of spot magnetic fields. The three components of the field strength in the cylindrical system of coordinates are

\[ B_r = B_0 J_1(kr) e^{-\beta z}, \]
\[ B_\theta = B_0 (r/R) J_1(kr) e^{-\beta z}, \]
\[ B_z = B_0 J_0(kr) e^{-\beta z}, \]  

where \( J_0 \) and \( J_1 \) are Bessel functions of the zeroth and first orders, respectively, and \( \beta \) is the scale-height of the magnetic field. Hagyard et al. (1977) took \( J_1(kR) = 0 \), so that \( kR = 3.8317 \).

The cylindrical coordinates can be transformed into Cartesian coordinates with the following formulae:

\[ B_x = B_r \cos \theta - B_\theta \sin \theta, \]
\[ B_y = B_r \sin \theta + B_\theta \cos \theta. \]  

Then the three parameters of the magnetic vector may be readily found to be

\[ B = B_0 e^{-\beta z} \sqrt{J_1^2(kr) \left( 1 + \frac{r^2}{R^2} \right) + J_0^2(kr)}, \]

\[ \gamma = \tan^{-1} \left[ \frac{J_1(kr)}{J_0(kr)} \sqrt{1 + \frac{r^2}{R^2}} \right], \]

\[ \chi = \tan^{-1} \frac{R \sin \theta + r \cos \theta}{R \cos \theta - r \sin \theta}. \]  

As a first approximation, we considered the case \( \beta = 0 \). In reality, magnetic fields in sunspots change with height, so that \( \beta \neq 0 \). The influence of the field gradient on the monochromatic images in Stokes parameters will be investigated in our future work.

It has to be pointed out that as the fan-shaped model exhibits circular symmetry, it is a radial model. On the other hand, as noted by Hagyard et al. (1977), in the model represented by Equations (10) the lines of force of the transverse field describe an Archimedean spiral given by \( r = R \theta \). It is then a spiral model.

### 4. Û Monochromatic Images and Models of Sunspot Magnetic Fields

Using the formulae in Section 2 and adopting consecutively the two models of spot magnetic fields described in Section 3, we have obtained numerical solutions of the Unno–Beckers equations (1) for the line \( \lambda 5250.216 \) and calculated the linear polari-
zation signal $\tilde{U}$. (\tilde{Q} can also be computed in this way. According to Landi Degl’Innocenti (1976), these two parameters have only a difference of 45° in their $\delta$ angles. Therefore, the $\tilde{Q}$ and $\tilde{U}$ monochromatic images should be similar to each other.) After making calculations for a series of points in a circular spot, its monochromatic $\tilde{U}$ images may be constructed. In order to learn the general features of such a monochromatic image, it is sufficient to perform our computations for a quarter of the circular spot. The observed monochromatic images published in the works of Hagyard et al. (1977, 1983) give only the geometrical shape of the positive and negative $\tilde{Q}$ (or $\tilde{U}$). Therefore, although we have obtained the numerical values of $\tilde{U}$, only their signs are used in this section. The four monochromatic images in Figure 1 show the results of computations for the radial model of sunspot magnetic fields given by formulae (8) and (9) with $B_0 = 1000, 2000, 3000, \text{and} 4000 \text{ G}$, respectively. On the $X$- and $Y$-axes, $\delta = 0^\circ$ and $90^\circ$, respectively. This figure clearly shows that a radial magnetic field can give rise to spiral $\tilde{U}$ monochromatic images. This confirms Landi Degl’Innocenti’s (1979) statement that Hagyard et al. (1977) were wrong in ascertaining that spiral magnetic fields should be responsible for the spiral $\tilde{Q}$ and $\tilde{U}$ images. It is not difficult to understand the physical reason of this fact. The MSFC vector magnetograph uses a narrow passband filter and observations are made with the filter positioned in the central part of a magnetoo-sensitive line. Under the action of magneto-optical effects, the linear polarization equivalent width of the central part of the magneto-sensitive line changes rapidly with the azimuthal angle and it becomes alternately positive and negative. This leads to the spiral configuration in $\tilde{U}$ and $\tilde{Q}$ monochromatic images. Figure 1 in our previous paper (Ye Shi-hui et al., 1983) shows that the interpretation of Landi Degl’Innocenti (1979) is correct.

Now we proceed with further computations assuming the Hagyard and West spiral model represented by Equations (10). Some of the monochromatic $\tilde{U}$ images thus obtained are shown in Figure 2 and they are drawn in polar coordinates. The azimuthal angle $\chi$ now coincides with polar angle. These images also exhibit some spiral configuration, but they are much more complicated than those in Figure 1. The monochromatic $\tilde{U}$ images obtained by means of the two models have the following differences: (1) the dividing lines between positive and negative regions in the images given by the fan-shaped model are more regular; (2) in the images yielded by the fan-shaped model, the dividing lines exhibit systematic displacements with varying $B_0$; (3) such images are closer to those observed than the images given by the spiral model. Therefore, we may conclude that the fan-shaped model of sunspot magnetic fields is closer to reality.

5. Absolute Intensity Distribution in $\tilde{U}$ Monochromatic Images

As we have already explained, only the signs of the calculated $\tilde{U}$'s are used in Figures 1 and 2. But our computations yield values of $\tilde{U}$ and $\tilde{Q}$ in units of intensity of the neighboring continuum. It is interesting to see how they depend on the positions within a spot and how they change with the field strength as well as the azimuthal angle. It is readily seen from Figure 3 that the radial model of spot magnetic fields gives rise to
Fig. 1a–d. Theoretical monochromatic $\tilde{U}$ images given by the radial model.
Fig. 1c.

$B_0 = 3000$ G

Fig. 1d.

$B_0 = 4000$ G
Fig. 2a–d. Theoretical monochromatic $\bar{U}$ images given by Hagyard et al.'s spiral model.
$B_o = 3000 \text{ G}$

Fig. 2c.

$B_o = 4000 \text{ G}$

Fig. 2d.
Fig. 3. Dependence of $\bar{U}$ on $\rho$ for the radial model.
Fig. 4. Relation between $\bar{U}$ at $\rho = 0.5$ and $\chi$ for the radial model.

Fig. 5. Relation between $\bar{U}$ at $\rho = 0.5$ and $B$ for the radial model.
$ar{U}(\rho)$ relations which are quite regular and may be represented by smooth curves. When only the $ar{U}$'s at $\rho = 0.5$ are considered, their relations with $\chi$ and $B$ also exhibit some regularities (see Figures 4 and 5). However, when Hagyard et al.'s spiral model is adopted, our calculations lead to $ar{U}(\rho)$ relations which seem to be quite irregular. Figure 6 is an example. It is at present difficult to say whether or how the above relations coincide with observations. Yet it is clear that different models of magnetic fields give rise to quite dissimilar $\bar{U}$ and $\bar{Q}$ monochromatic images and that by a comparison of calculated and observed images one may learn which model is closer to reality. This is in need of further investigation.

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References