Non-linear structures in self-gravitating disks

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Abstract. The non-linear development and the evolution of a perturbation in self-gravitating disks are investigated on the basis of wave-wave interaction. It is shown that the perturbation is unstable with respect to self-modulation and self-focusing. The analysis is applied to spiral structure in spiral galaxies and to periodically spaced fragmentation of filaments in molecular clouds.

Key words: hydrodynamics – instabilities – ISM: clouds – galaxies spiral

1. Introduction

A recent report by Dutrey et al. (1991) has shown the existence of fragments regularly spaced with a separation of about one parsec along the main filament of the Orion A molecular cloud. Observations show that this molecular cloud consists of sheets or filaments in which the fragments are embedded (Bally et al. 1991). This is also the appearance of spiral patterns in galaxies. As far as we understand them, these large-scale structures are self-organization phenomena due to non-linear interactions in disk-like gravitating systems. In a non-linear theory, disturbances of this type are unstable to self-focusing and self-modulation; due to these instabilities, an initially uniform disturbance can undergo spatial modulation. Therefore, it is possible to form regular patterns of density, provided that the perturbation amplitudes are finite.

The non-linear development and evolution of a density perturbation in a fluid disk with rotation and self-gravitation can be described in an Hamiltonian approach (Zakharov 1971). Such a method allows to express the non-linear terms of the equations as expansions with respect to powers of the wave amplitude. Under the assumption of a narrow wave packet and after consideration of the non-linear wave-wave interactions up to the fourth order, we derive in this paper a non-linear equation for the envelope. Solving this equation, we obtain a periodic solution, which means that the wave amplitude is self-modulated and has periodic maxima and minima (Sect. 2).

In Sect. 3 we discuss two applications of self-modulated density waves in the disk. One application is to the observed periodically spaced fragmentation embedded in filaments or density clumps in molecular clouds. Another application is to spiral galaxies.

In the appendix we present some formulas which are needed in calculating some quantities in the paper.

2. The wave-wave interaction in a disk

Consider an infinitesimally thin gravitating fluid layer rotating uniformly at an angular velocity \(\Omega/Z\). In the frame of reference rotating at \(\Omega\), the hydrodynamical equations may be written,

\[
\frac{\partial \sigma}{\partial t} + \nabla \cdot (\sigma v) = 0 ,
\]

\[
\frac{\partial v}{\partial t} + (v \cdot \nabla) v + 2 \Omega \times v = -\frac{1}{\sigma} \nabla p + g + \Omega^2 r ,
\]

\[
g = -\nabla \Phi , \quad \nabla \times g = 0 ,
\]

\[
\nabla^2 \Phi + \frac{\partial^2 \Phi}{\partial Z^2} = 4 \pi G \sigma \delta(Z) ,
\]

where \(\sigma\) is the surface density, \(\Phi\) the gravitational potential; all vector values and operators refer to the plane \((x, y)\).

The unperturbed state \((v = 0)\) satisfies the following equations:

\[
\Omega^2 r = \nabla \Phi_0 + \frac{1}{\sigma_0} + \nabla P_0 ,
\]

\[
\nabla^2 \Phi_0 + \frac{\partial^2 \Phi_0}{\partial Z^2} = 4 \pi G \sigma_0 \delta(Z) .
\]

Using Green's function we can immediately write the potential in the form \((Z = 0)\)

\[
\Phi = -G \int \frac{\sigma(r')}{|r - r'|} \, dr' .
\]

The equation of state is

\[
P = c \sigma^\gamma
\]
where the pressure $P$ is assumed to act only in the plane of the
disk. Writing

$$\sigma = \sigma_0 + \tau, \quad \left( \frac{\tau}{\sigma_0} \right) \ll 1,$$

(2.8)

where variable $\tau$ describes the density wave on the background
of the unperturbed layer, and using the equilibrium Eq. (2.4),
the right side of Eq. (2.2) becomes

$$-\nabla \left\{ \frac{\gamma}{\gamma - 1} c_s((\sigma_0 + \tau)^{\gamma - 1} - \sigma_0^{\gamma - 1}) \right\}$$

$$+ \nabla \left[ G \int \frac{\tau(r')}{|r - r'|} dr' \right];$$

hence Eq. (2.2) has the form

$$\frac{\partial \nu}{\partial t} + (v \cdot \nabla) v + 2 \Omega \times v = -\nabla \frac{\delta E}{\delta \tau},$$

(2.9)

where

$$E = \frac{G}{2} \int \frac{\tau(r) \tau(r')}{|r - r'|} dr dr'$$

$$+ \frac{1}{2} c_s^2 \int \frac{\tau^2}{\sigma_0^2} + \frac{2}{3} \frac{\tau^3}{\sigma_0^3} + \ldots,$$

(2.10)

in which $c_s = (\gamma \sigma_0^{\gamma - 1})^{1/2}$ is the sound speed. $E$ contains only
terms of the second and higher powers of $\tau$, without constants
and subintegral terms.

We immediately obtain from Eqs. (2.1) and (2.9), with the aid of the Gaussian theorem,

$$\partial / \partial t \mathcal{H} = 0,$$

(2.11a)

where the Hamiltonian is

$$\mathcal{H} = \int \frac{1}{2} \sigma v^2 dr + E.$$

(2.11b)

We should point out that the term corresponding to the centripetal force in Eq. (2.2) has been disregarded in the study of Churilov & Shukhman (1981), because the term might result in a divergent form in (2.11) according to their calculation (Fridman & Polyachenko 1984); but in fact the divergent term may not occur, as our above derivation shows.

Linearizing Eqs. (2.1), (2.2) and (2.3) and assuming that
the unperturbed density $\sigma_0$ is slowly varying in space
with respect to wave fields, and using the WKB approximation,
k $\gg \partial / \partial r \ln |r|$, where $k$ is the wave number of the perturbation

$$\tau \propto \tilde{r} \exp(-i\omega t + ik \cdot r),$$

one can get the following dispersion relation

$$\omega_k^2 = 4\Omega^2 + k^2 c_s^2 - 2\pi G\sigma_0 k,$$

(2.12)

In the majority of cases the Hamiltonian (2.11b) coincides in its physical sense with the wave energy in the medium, and

the dispersion properties of the wave in the linear approximation
must be such that the square of the frequency is

$$\omega_k^2 \geq 0,$$

which turns out to be the stability of the disk: one has from
(2.12) (Binney & Tremaine 1987)

$$Q \equiv 2c_s\Omega / \pi G\sigma_0 \geq 1.$$

Now we bring the original formulas into an Hamiltonian
description by introducing

$$v = \frac{\lambda}{\sigma} \nabla \mu + \nabla \varphi - \Omega \times r,$$

(2.13)

where $\lambda$ and $\mu$ are the Clebsch variables suitable for barotropic fluids. With these variables, the Eqs. (2.1) and (2.2) take the following forms (Zakharov et al. 1985)

$$\frac{\partial \tau}{\partial t} = \frac{\delta \mathcal{H}}{\delta \varphi}, \quad \frac{\partial \varphi}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \tau},$$

$$\frac{\partial \lambda}{\partial t} = \frac{\delta \mathcal{H}}{\delta \mu}, \quad \frac{\partial \mu}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \lambda}.$$

(2.14)

Because $v$ in (2.13) is an explicit function of coordinates, we need an additional canonical transformation which will be done in two stages (Churilov & Shukhman 1981). First, put

$$\lambda = \sqrt{\frac{\sigma}{2}} (\lambda' + \mu'), \quad \mu = \frac{1}{\sqrt{2\sigma}} (\mu' - \lambda'),$$

$$\varphi = \varphi' + \frac{\lambda'^2 - \mu'^2}{4\sigma},$$

and then eliminate $\Omega \times r$ in $v$ by

$$\lambda' = \lambda'' + \sqrt{2\Omega \sigma} y, \quad \mu' = \mu'' - \sqrt{2\Omega \sigma} x,$$

$$\varphi' = \varphi'' - \sqrt{\frac{\Omega}{2\sigma}} (x \lambda'' + y \mu'').$$

The form of Hamiltonian equations is not changed using the new canonical variables and $v$ becomes

$$v = \lambda'' \nabla \mu'' - \mu'' \nabla \lambda'' - \sqrt{\frac{2\Omega}{\sigma}} (\lambda'' e_x + \mu'' e_y) + \nabla \varphi''$$

where $e_x$ and $e_y$ are the unit vectors on the $x$ and $y$ axes. In the following we omit the primes on $\lambda''$, $\mu''$, $\varphi''$.

It is not difficult to confirm that the Fourier components of canonical variables are also canonical, so we have

$$\frac{\partial \tau_k}{\partial t} = \frac{\delta \mathcal{H}}{\delta \tau_k}, \quad \frac{\partial \varphi_k}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \tau_k},$$

$$\frac{\partial \lambda_k}{\partial t} = \frac{\delta \mathcal{H}}{\delta \mu_k}, \quad \frac{\partial \mu_k}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \lambda_k}.$$
for the corresponding Fourier components $\tau_k$, $\varphi_k$, $\lambda_k$, and $\mu_k$. Introducing further the canonical transformations
\begin{align}
\tau_k &= \frac{k \sigma_0^{1/2}}{(2\omega_k)^{1/2}} (a_k + a_*^{-1} k) , \\
\varphi_k &= -i \frac{\omega_k^2 - 4 \Omega_k^2}{(2 \sigma_0)^{1/2} k \omega_k} (a_k - a_*^{-1} k) , \\
\mu_k &= -\frac{\Omega_k^{1/2}}{k \omega_k} \left\{ (2 \Omega_k y_i - i \omega_k k_x) a_k \\
&- (2 \Omega_k y_i + i \omega_k k_x) a_*^{-1} k \right\}, \\
\lambda_k &= -\frac{\Omega_k^{1/2}}{k \omega_k} \left\{ (2 \Omega_k x_i + i \omega_k k_y) a_k \\
&- (2 \Omega_k x_i - i \omega_k k_y) a_*^{-1} k \right\},
\end{align}
the previous equations led to a Hamiltonian description (Churilov & Shkunov 1981)
\begin{equation}
\frac{\partial a_k}{\partial t} = -i \frac{\partial \mathcal{H}}{\partial \varphi_k} .
\end{equation}
In this description the quadratic term in the Hamiltonian $\mathcal{H}$ is
\begin{equation}
\mathcal{H}^{(2)} = \int \omega_k a_k a_*^{*} d\mathbf{k}
\end{equation}
which represents total wave energy with a density of the number of "quanta" $a_k a_*^{-1} k$ and a "quantum" energy $\hbar \omega_k$.

The third- and fourth-order terms in the expansion of the Hamiltonian $\mathcal{H}$ in powers of the variable $a_k$ are respectively:
\begin{align}
\mathcal{H}^{(3)} &= \int \left[ \frac{1}{3} V_{kk,k_2} a_k a_k a_{k_2} \delta(k + k_1 + k_2) \\
&+ V_{kk,k_1,k_2} a_k a_k a_{k_2} \delta(k - k_1 - k_2) + \text{c.c.} \right] d\mathbf{k}, \\
\mathcal{H}^{(4)} &= \int \left[ 4W_{kk,k_1,k_2,k_3} a_k a_k a_{k_1} a_{k_2} a_{k_3} \delta(k + k_1 + k_2 + k_3) \\
&+ 3W_{kk,k_1,k_2,k_3} a_k a_k a_{k_1} a_{k_2} a_{k_3} \delta(k + k_1 + k_2 - k_3) \\
&+ \text{c.c.} \right] d\mathbf{k}.
\end{align}
The quantities $V_{kk,k_1}, W_{kk,k_1,k_2,k_3}, \text{etc.}$ represent the matrix elements of different types of wave interaction. These matrix elements can be calculated and we give their explicit expressions in the appendix.

The equation (2.16) up to the fourth order term $\mathcal{H}^{(4)}$ gives
\begin{equation}
\frac{\partial a_k}{\partial t} + i \omega_k a_k = -i \int d\mathbf{k} \int d\mathbf{k}_2 [V_{kk,k_1} a_k a_k a_{k_2} \delta(k + k_1 - k_2) \\
+ 2V_{kk,k_1} a_k a_{k_1} a_{k_2} \delta(k + k_1 - k_2) \\
+ V_{kk,k_1} a_k a_{k_1} a_{k_2} \delta(k + k_1 + k_2)] \\
-i \int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 [4W_{kk,k_1,k_2,k_3} a_k a_k a_{k_2} a_{k_3} \delta(k - k_1 - k_2 - k_3) \\
+ 6W_{kk,k_1,k_2,k_3} a_k a_k a_{k_1} a_{k_2} a_{k_3} \delta(k + k_1 - k_2 - k_3) \\
+ 12W_{kk,k_1,k_2,k_3} a_k a_k a_{k_1} a_{k_2} a_{k_3} \delta(k - k_1 + k_2 + k_3) \\
+ 4W_{kk,k_1,k_2,k_3} a_k a_k a_{k_1} a_{k_2} a_{k_3} \delta(k + k_1 + k_2 + k_3)].
\end{equation}
If there are only waves such that the resonance conditions for the three wave processes
\begin{equation}
\omega_k + \omega_{k_2} = \omega_k , \quad k = k_1 + k_2 ; \\
\omega_k + \omega_{k_2} + \omega_{k_3} = 0 , \quad k + k_1 + k_2 = 0
\end{equation}
are not satisfied, the three-wave interaction is not effective and it is necessary to consider the four-wave interaction.

Dividing $a_k$ into a fast oscillating part $f_k$ and a slowly varying one $S_k$:
\begin{equation}
a_k = (f_k + S_k) e^{-i \omega_k t}
\end{equation}
where
\begin{equation}
|f_k| \ll |S_k|
\end{equation}
and restricting to the fast oscillating terms in Eq. (2.20), we obtain
\begin{align}
\frac{\partial f_k}{\partial t} &= -i \int d\mathbf{k}_1 d\mathbf{k}_2 \left\{ V_{kk,k_1} S_k S_{k_1} S_{k_2} \exp[i(\omega_k - \omega_{k_1} - \omega_{k_2}) t] \\
&\times \delta(k - k_1 - k_2) + 2V_{kk,k_1} S_{k_1}^* S_{k_2} \exp[i(\omega_k + \omega_{k_1} - \omega_{k_2}) t] \\
&\times \delta(k + k_1 - k_2) \right\}.
\end{align}
Owing to the slow variation of $S_k$, $f_k$ in above equation can be integrated
\begin{align}
f_k &= -\int d\mathbf{k}_1 d\mathbf{k}_2 \left\{ V_{kk,k_1} S_k S_{k_1} S_{k_2} \exp[i(\omega_k - \omega_{k_1} - \omega_{k_2}) t] \\
&\times \delta(k - k_1 - k_2) + 2V_{kk,k_1} S_{k_1}^* S_{k_2} \exp[i(\omega_k + \omega_{k_1} - \omega_{k_2}) t] \\
&\times \delta(k + k_1 - k_2) \right\}.
\end{align}
Substituting $f_k$ into the slowest part of Eq. (2.20), we have
\begin{equation}
\frac{\partial S_k}{\partial t} = -i \int T_{kk,k_1,k_2,k_3} S_{k_1} S_{k_2} S_{k_3} \exp[i(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) t] \\
\times \delta(k + k_1 - k_2 - k_3) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3
\end{equation}
where
\begin{align}
T_{kk,k_1,k_2,k_3} &= -2V_{kk,k_1,k_2,k_3} V_{(k_1 - k_2)k_1} S_{k_1} S_{k_2} S_{k_3} \\
&\quad \quad \frac{1}{\omega_{k_1} + \omega_{k_2} + \omega_{k_3}} \\
&\quad - 2V_{kk,k_1,k_2,k_3} S_{k_1}^* S_{k_2} S_{k_3} \frac{1}{\omega_{k_1} + \omega_{k_2} - \omega_{k_3}} \\
&\quad - 2V_{kk,k_1,k_2,k_3} S_{k_1}^* S_{k_2} S_{k_3} \frac{1}{\omega_{k_1} + \omega_{k_2} + \omega_{k_3}} \\
&\quad + 6W_{kk,k_1,k_2,k_3} S_{k_1}^* S_{k_2} S_{k_3} \\
&\quad + 4W_{kk,k_1,k_2,k_3} S_{k_1}^* S_{k_2} S_{k_3} + 4W_{kk,k_1,k_2,k_3} S_{k_1}^* S_{k_2} S_{k_3}.
\end{align}
Consider a narrow wave packet
\begin{equation}
q = k - k_0 , \quad |q| \ll |k_0|
\end{equation}
the frequency $\omega_k$ may be expanded in the vicinity of the $k_0$:

$$\omega_k = \omega_{k_0} + q \cdot 
\left( \frac{\partial \omega_k}{\partial k} \right)_{k_0} + \frac{1}{2} \left( \frac{\partial^2 \omega_k}{\partial k \partial k} \right)_{k_0} q_q q_t ;$$

(225b) taking

$$A_k = S_k \exp \left\{ -i \left[ q \cdot \left( \frac{\partial \omega_k}{\partial k} \right)_{k_0} + \frac{1}{2} \left( \frac{\partial^2 \omega_k}{\partial k \partial k} \right)_{k_0} q_q q_t \right] \right\} \equiv S_k e^{-i(\omega_{k_0} - \omega_k)t} ;$$

(26a)

we get from (223)

$$\frac{\partial A_k}{\partial t} + i \left[ \mathbf{v}_q \cdot d + \frac{1}{2} D_{ij} q_i q_j \right] A_k = -i T \int A_{k_1} A_{k_2} A_{k_3} \delta(k + k_1 - k_2 - k_3) \, dk_1 \, dk_2 \, dk_3$$

(26b) where

$$\mathbf{v}_q = \left( \frac{\partial \omega_k}{\partial k} \right)_{k_0} , \quad D_{ij} \equiv \left( \frac{\partial^2 \omega_k}{\partial k_i \partial k_j} \right)_{k_0} , \quad T = T_{k_0 k_1 k_2 k_3} .$$

For an isotropic medium, $\omega_k = \omega(k)$, one has

$$D_{ij} = v' k_0 k_1 k_2 + \frac{v_9}{k_0} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) ;$$

then

$$D_{ij} q_i q_j = \frac{v_9}{k_0} q_1^2 + v_9 q_1^2$$

where

$$v_9 = \left( \frac{\partial v_9}{\partial k} \right)_{k_0} , \quad q_1 = \frac{q \cdot k_0}{k_0} , \quad q_1^2 = q^2 - q_1^2 .$$

Multiplying (227) by $e^{i\sigma r}$ and integrating over $q$ yield

$$i \left( \frac{\partial A}{\partial t} + (v_q \cdot \nabla) A \right) + \frac{1}{2} v_9 \nabla_1^2 A + \frac{1}{2} v_9' \nabla_1^2 A$$

$$-(2\pi)^2 \tau|A|^2 = 0 ;$$

(228) where

$$A(r, t) = \frac{1}{(2\pi)} \int A_q e^{i\sigma r} dq .$$

Besides, it follows from (222) and (226b)

$$a_k \approx S_k e^{-i\omega_{k_0}t} ;$$

then one has from (2.15a)

$$\tau = \int \frac{k e^{i\omega_{k_0}t} dk}{\sqrt{2\omega_k}}$$

$$= \frac{1}{2} \sqrt{2\omega_k} a_k e^{i\omega_{k_0}t} dk = \text{Re} \left\{ 2\sqrt{2\omega_k} \int \frac{k}{\sqrt{2\omega_k}} a_k e^{i\omega_{k_0}t} dk \right\}$$

(229a) where

$$\tilde{\tau} = 2\sqrt{\sigma_0} \int \frac{k}{\sqrt{2\omega_k}} A_k e^{i(k - r - k_0) + i\omega_{k_0}t} e^{-i\omega_{k_0}t} \frac{dk}{(2\pi)}$$

$$A(r, t) = \frac{1}{\sqrt{2\omega_k}} \int \frac{k e^{i\omega_{k_0}t} dk}{\sqrt{2\omega_k}} A_k e^{i(k - r - k_0) + i\omega_{k_0}t}$$

$$\approx (r, t) \sqrt{2\omega_k} \frac{e^{i(r - \omega_{k_0}t)} \frac{dk}{\sqrt{2\omega_k}}}{\sqrt{2\omega_k}} .$$

(229b)

If the $x$-axis is chosen to be in the direction of $k_0$, (2.28) becomes

$$i \left( \frac{\partial A}{\partial t} + v_9 \frac{\partial A}{\partial x} \right) + \frac{1}{2} v_9' \frac{\partial^2 A}{\partial x^2} + \frac{1}{2} k_0 \nabla_1^2 A$$

$$-(2\pi)^2 \tau|A|^2 = 0 .$$

(230)

Equation (2.31) can be simplified in the following way. If the envelope $A$ is represented as

$$A = A(\xi), \quad \xi = kr - m\theta ,$$

(232)

(2.31) becomes

$$i \left( \frac{\partial A}{\partial t} + v_9 k \frac{\partial A}{\partial \xi} \right) + \frac{1}{2} v_9' k^2 \frac{\partial^2 A}{\partial \xi^2} + \frac{1}{2} k_0 \frac{m^2}{r^2} \frac{\partial^2 A}{\partial \xi^2}$$

$$-(2\pi)^2 \tau|A|^2 = 0 .$$

(233)

If $k_0$ is in the vicinity of the minimum of the dispersion curve for theses waves are excited easily, then $v_9/k_0 v_9' \ll 1$; further, we use the WKB approximation implying $m^2/k^2 r^2 \ll 1$; after these considerations, we can omit the fourth term in Eq. (2.33) with respect to the third one:

$$i \left( \frac{\partial A}{\partial t} + v_9 k \frac{\partial A}{\partial \xi} \right) + \frac{1}{2} v_9' k^2 \frac{\partial^2 A}{\partial \xi^2} - (2\pi)^2 \tau|A|^2 = 0 .$$

(234)
3. Application and discussion

3.1. Model of the filaments in molecular clouds

We suggest that the periodically spaced fragmentation in Orion A can be described as the effect of nonlinear self-focusing and self-modulation in a gravitating disk by Eq. (2.30).

Put $A = \psi e^{i\theta}$ and separating (2.30) into real and imaginary parts, one has

$$\frac{\partial \theta}{\partial t} + v_k \frac{\partial \theta}{\partial x} + \frac{1}{2} v_k^2 \left( \frac{\partial \theta}{\partial x} \right)^2 + \frac{v_k}{k_0} (\nabla_\perp \theta)^2 + \frac{(2\pi)^2 T}{2} \psi^2$$

$$- \frac{v_k^2}{2} \psi \frac{\partial^2 \psi}{\partial x^2} - \frac{v_k}{k_0} \nabla_\perp \psi = 0 ,$$

(3.1a)

$$\frac{\partial \psi^2}{\partial t} + v_k \frac{\partial \psi^2}{\partial x} + v_k \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial x} \right) + \frac{v_k}{k_0} \nabla_\perp (\psi^2 \nabla_\perp \theta) = 0 .$$

(3.1b)

We consider an initial wave specified by

$$\psi = \psi_0 e^{-i\theta_0 t}, \quad \theta_0 = (2\pi^2 T)^{1/2} \psi_0^2 ,$$

(3.2)

which is an exact solution of Eq. (3.1) and perturbations

$$\psi_1 = \psi - \psi_0 , \quad \theta_1 = \theta - \theta_0 , \quad (\theta_0 = -\theta_0 t) ,$$

(3.3a)

$$(\psi_1, \theta_1) \propto \exp[-i(\Omega t + i\kappa_1 r)] ;$$

(3.3b)

substituting expressions (3.2) and (3.3) into Eq. (3.1) and linearizing with respect to $\psi_1$ and $\theta_1$, we find (Li 1987, Sect. 31)

$$\bar{\Omega} = \bar{k}_1 v_g \pm \sqrt{L(L + 2((2\pi^2 T)^2) \psi_0^2}$$

(3.4)

where

$$L = \frac{v_k^2}{2} \frac{k_0^2}{k_0^2} + \frac{1}{2} v_k^2 \frac{L_2}{v_k^2} .$$

(3.5)

Thus, an instability occurs when and only when

$$\bar{k}_1^2 < \frac{1}{2} 4k_0^2 \psi_0^2 (-(2\pi^2 T)^2) , \quad (T < 0) ,$$

(3.6)

provided that

$$\bar{k}_1 \gg \left( \frac{k_0 \psi_0^2}{v_g} \right)^{1/2} .$$

(3.7)

The condition (3.6) may be rewritten as

$$\bar{k}_1 \equiv \frac{2\pi}{k_1} > \bar{\lambda}_c ,$$

(3.8a)

where

$$\bar{\lambda}_c = \frac{\pi}{(k_0 \psi_0^2)^{1/2}} ,$$

(3.8b)

$$\beta_0 = \frac{-(2\pi^2 T)^2}{(\nu_g \psi_0^2)} ,$$

(3.8c)

Hence the self-modulation could lead to the division of the wave into localized structures of typical scale $l$, such that $l \sim \bar{\lambda}_c$. In this direction, bunching will proceed until the profile is approximately that of a one-dimensional stable structure (fer Haar & Tsyтович 1981), that is, a soliton which is the solution of the one-dimensional variant of Eq. (2.18) (Li 1987, Sect. 28):

$$\ddot{A}(y) = \psi_0 \operatorname{sech} \left( \sqrt{\beta_0 \psi_0^2} y \right) e^{\frac{i}{\kappa} \beta y \psi_0^2 t + \phi_0} .$$

As can be expected, the half-width of the solution is the scale $\bar{\lambda}_c$ for self-modulation.

In a coordinate system moving with velocity $v_g$, Eq. (2.18) becomes

$$i \frac{\partial A}{\partial t} + \frac{1}{2} \frac{\partial^2 A}{\partial x^2} + \frac{1}{2} \nabla_\perp A + \frac{\beta |A|^2}{2} A = 0 ,$$

(3.9)

where

$$\tau = v_g t , \quad (\nabla_\perp = \frac{k_0 v_g^2}{v_g} )^{-1/2} \nabla_\perp , \quad \tilde{x} = x - v_g t ,$$

(3.10a)

$$\beta = \frac{-(2\pi^2 T)^2}{v_g^2} .$$

(3.10b)

Now let us find solutions of Eq. (3.9) by looking for the solutions which are of the shape of the soliton along the $y$-direction. We make an ansatz for the solutions in the form

$$A = \psi(x, \tau) \operatorname{sech} \left( \sqrt{\beta \psi_0^2} (\tilde{y} - \tilde{y}_0) \right) e^{i\theta(x, \tau)}$$

(3.11)

in which $\psi$ and $\theta$ are also weak-dependent functions on $\tilde{y}$. Substituting (3.11) into (3.9) and multiplying by $d \tilde{y}/2\pi$ and integrating over the $\tilde{y}$-coordinate from $(-\lambda_c)$ to $(+\lambda_c)$ coordinate from $(-\lambda_c)$ to $(+\lambda_c)$, after simple manipulation, Eq. (3.9) becomes

$$\frac{\partial \theta}{\partial \tau} + \left( \frac{\partial \theta}{\partial x} \right)^2 - \beta \left( \psi^2 - \frac{1}{2} \psi_0^2 \right) - \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} = 0 ,$$

(3.12a)

$$\frac{\partial \psi^2}{\partial \tau} + \frac{\partial}{\partial \tilde{x}} \left( \psi^2 \frac{\partial \theta}{\partial \tilde{x}} \right) = 0 .$$

(3.12b)

For seeking a solution in the form

$$\psi = \psi(x - u_\tau \tau) , \quad \theta = \theta(x - u_\tau \tau) ,$$

(3.13)

obviously Eq. (3.12b) has an integral,

$$\theta = u_\tau [(\tilde{x} - \tilde{x}_0) - u_\tau \tau] .$$

(3.14)

Then Eq. (3.12) becomes

$$\frac{1}{2} \left( \frac{d \psi}{d \tilde{x}} \right)^2 + p(\psi) = c$$

(3.14)

where

$$p(\psi) = \frac{1}{2} \beta \psi^4 - \alpha \psi^2 ,$$

(3.16a)

$$\alpha = u_\tau^2 + \frac{1}{2} \beta \psi_0^2 - u_\tau u_e$$

(3.16b)

and $c$ is the integration constant. If we interpret $\psi$ and $\tilde{x}$ as space and time coordinates respectively, we may regard Eq. (3.15) as
the energy equation for the motion of a particle. Obviously under the
condition,
\[ c < 0 \]  
(3.17)
Equation (3.15) represents a periodic motion oscillating between two consecutive real zeros of \((c - p(p))\), which describe the self-focusing phenomenon. When
\[ \alpha > 0 , \quad -\frac{\alpha^2}{2\beta} < c < 0 \]  
(3.18)
Equation (3.15) on integration yields
\[ \psi = \psi_2^0 \{ 1 - (1 - q^{-2})c_n^2 [\beta^{1/2} \psi_2^0 (\bar{x} - \bar{x}_0 - u_e \tau) \}^{-1} \]  
(3.19)
where \(\psi_{1,2}^0\) are two real roots of the equation
\[ p(\psi) = c \]  
(3.20)
and \(c_n\) is the Jacobian Elliptic function with the modulus \(s_0\)
\[ s_0^2 \equiv 1 - q^2 \equiv 1 - \left(\frac{\psi_2^0}{\psi_2^0}\right)^2 < 1 \]  
(3.21)
From Eq. (2.29), we get
\[ \tau(\bar{r}) = \frac{2\omega_0 \psi_0^{1/2}}{(2\omega_0 \psi_0^{1/2}) \psi_0 (\bar{y} - \bar{y}_0)} \frac{\coth(\omega_0 t - k_0 \cdot \bar{r})}{\cos(\omega_0 t - k_0 \cdot \bar{r})} \]  
(3.22)
Then
\[ \psi_2^0 = \frac{(2\omega_0 \psi_0^{1/2})}{(2\omega_0 \psi_0^{1/2}) \psi_0^{1/2}} \tau_{\text{max}} \]  
(3.23)
where \(\tau_{\text{max}}\) is maximal density perturbation.
From Eq. (3.19), we find that the envelope has a period along \(0x\)
\[ \lambda_x = \frac{2k[s_0]}{\beta^{1/2} \psi_2^0} \]  
(3.24)
where \(k[s_0]\) is the complete Elliptic Integral of the first kind
\[ k[s_0] = \int_0^{\pi/2} \frac{d\phi}{[1 - (1 - q^2) \sin \phi]^{1/2}} \]  
(3.25)
For \(q \ll 1,\)
\[ k[s_0] \approx \ln \frac{4}{q} \]  
(3.26)
In the appendix we calculate \(\beta\) in Eq. (A13) as
\[ \beta = \frac{c^2 G \sigma_0 k_0}{c_3} \]  
(3.27)
Inserting (3.23), (3.27) into (3.24), we obtain
\[ \lambda_x = \left( \frac{1}{2} \omega^{1/2} \left( \frac{A}{\eta} \right) \frac{G \sigma_0 \tau_{\text{max}}}{c_3} \right)^{-1} \times 2k[s_0] \]  
(3.28)
where we have used the Eq. (A1) and Eq. (A2) in the appendix.

We assume that the reasonable values \(\sigma_0 \sim 0.01 \text{ g cm}^{-2}, \) \(c_3 \sim 2 \times 10^6 \text{ cm s}^{-1}, \tau_{\text{max}} / \sigma_0 \sim 0.2, Q \sim 1.2, \gamma = 3\) and choose \(\eta\) to be close to the 'most unstable wave number' \(k^* = \pi G \sigma_0 / c_3,\) for example, \(\eta = 0.6;\) the value of \(q\) in (3.26) is supposed to be \(1/10;\) as one may see, the true value of \(q\) is not very important because \(q\) is in the argument of a logarithmic function. Now (3.28) is calculated as
\[ \lambda_x \sim 1.2 \text{ pc} . \]  
(3.29)
This value is in good agreement with the observed separation of the cloud fragments in the filament which is reported to be 1 parsec to 1.5 parsecs.

The filaments of Orion A extend for many degrees but are only a few arc minutes wide; aspect ratios in excess of 30:1 are common (Bally et al. 1991). For a filament at rest \((\psi_2^0 \sim \psi_0^0),\) taking a length scale \(L_A = 15 \text{ pc}\) (Dutrey et al. 1991), we get for the degree of collimation of the filament
\[ \chi = \frac{2d}{L_A} \sim \frac{\lambda x}{L_A k[s_0]} H_0^1 \sim \frac{1}{50} , \]  
(3.30)
which is also in good agreement with the observation data.

3.2. Model of spiral structure

Now we consider the application of the nonlinear self-modulation theory to spiral galaxies. The starting point is Eq. (3.24) which we derived in a locally Cartesian system and in the WKB approximation. In a coordinate system moving with velocity \(v_\theta,\) Eq. (3.24) can be written as
\[ i \frac{\partial A}{\partial \tau} + \frac{1}{2} \frac{\partial^2 A}{\partial \xi^2} + \beta |A|^2 A = 0 \]  
(3.31)
where
\[ \tau = v_\theta t , \quad \xi = \frac{k}{k} \xi - v_\theta t , \quad \beta = \frac{-(2\pi)^2 T}{v_\theta^2} . \]  
(3.32)
As before, we separate Eq. (3.31) into real and imaginary parts
\[ \frac{\partial \theta}{\partial \tau} + \frac{1}{2} \left( \frac{\partial \theta}{\partial \xi} \right)^2 - \beta \psi^2 - \frac{1}{2} \frac{\partial^2 \psi}{\partial \xi^2} = 0 , \]  
(3.33)
\[ \frac{\partial \psi}{\partial \tau} + \frac{\partial^2 \psi}{\partial \xi^2} + \left( \psi^2 \frac{\partial \theta}{\partial \xi} \right) = 0 \]  
(3.34)
in which we have written \(A = \psi e^{i\theta} .\) Equation (3.33) is similar to Eq. (3.12), so we can find the solution in the same way. The only difference is in expression (3.16b). Now it is
\[ \alpha = \frac{1}{2} u_e^2 - u_{ue} . \]  
(3.35)
Therefore, using the results obtained in the previous part, we obtain the solution
\[ \psi = \psi_2^0 \{ 1 - (1 - q^{-2})c_n^2 [\beta^{1/2} \psi_2^0 (\bar{x} - \bar{x}_0 - u_e \tau)] \}^{-1/2} \]  
(3.36)
From (3.32), (3.32) and (3.35), it can be shown that due to nonlinear wave-wave interactions, the density waves are self-modulated and the amplitude has periodic maxima and minima. Now if the regions where the density wave amplitude is maximal are identified as the spiral arms, the spiral structure of galaxies is locally constructed. Due to this, the radial separation between adjacent arms should be half the period of the density wave amplitude for a two-armed spiral. Let us calculate the period. From (3.35), the period is expressed as in (3.24):

\[
\lambda_r = \frac{2k[s_0]}{\beta^{1/2} \psi^9_2}
\]

and from (2.29), we get the same equation (3.23) which relates the wave amplitude to the density contrast. Thus, the expression of the period is also the same as (3.28)

\[
\lambda_r = \left( \frac{1}{2} \frac{a^{1/2}}{c^3} \right) \left( \frac{A}{\eta} \right)^{1/2} \frac{G \sigma_0}{c^2} \frac{\tau_{\text{max}}}{\sigma_0} \frac{1}{2} k[s_0]
\]

Using the formulas in the appendix and taking \( \sigma_0 \approx 50 \, M_\odot \, \text{pc}^{-2} \), \( c_s \approx 10 \, \text{km s}^{-1} \), \( Q \approx 1.2 \), \( \gamma \approx 3 \), \( \tau_{\text{max}}/\sigma_0 \approx 0.15 \), \( \eta \approx 0.6 \), \( q \approx 1/10 \), we find \( \lambda_r \approx 3.6 \, \text{kpc} \). Then, the radial separation between adjacent arms is about \( \lambda_r/2 \approx 1.8 \, \text{kpc} \).

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Appendix

In this appendix we will find the expression for \( \beta \) which is needed in formula (3.24).

The quantities \( Q, \eta, \kappa \) are defined as follows

\[
\Omega = \frac{Q \pi G \sigma_0}{2c_s}, \quad k_0 = \frac{2 \pi G \sigma_0}{c_s^2}, \quad \kappa = 2 \Omega, \quad \text{(A1)}
\]

then

\[
\omega_{k_0} = (4Q^2 - 2 \pi G \sigma_0 k_0 + k_0^2 c_s^2)^{1/2} = (Q^2 - 4 \eta + 4 \eta^2)^{1/2} \frac{\pi G \sigma_0}{c_s} \equiv A_0 \left( \pi G \sigma_0 / c_s \right), \quad \text{(A2)}
\]

\[
\omega_{2k_0} = (4Q^2 - 2 \pi G \sigma_0 \times 2k_0 + 4k_0^2 c_s^2)^{1/2} = (Q^2 - 8 \eta + 16 \eta^2)^{1/2} \frac{\pi G \sigma_0}{c_s} \equiv B_0 \left( \pi G \sigma_0 / c_s \right), \quad \text{(A3)}
\]

From the following formulas for the coefficients \( V \) in Eq. (2.24) (Fridman & Polyachenko 1984),

\[
V_{kk_1k_2} = U_{kk_1k_2} \sigma_0 / \sqrt{4 \pi N_{kk_1k_2}}, \quad \text{(A4)}
\]

\[
U_{kk_1k_2} = \frac{2}{\sqrt{4 \pi N_{kk_1k_2}}} \left\{ 2 \omega_{kk_1k_2} \omega_{kk_1k_2} \kappa^2 (k \times k_2)^2 + 2 \kappa \omega_{kk_1k_2} (k \times k_1) \cdot (k \times k_2) \right\} + 2 \omega_{kk_1k_2} \omega_{kk_1k_2}^2 (k_1 \cdot k_2) + \frac{2}{3} (\gamma - 2) \kappa^3 k_1^2 k_2^2 \omega_{kk_1k_2} \omega_{kk_1k_2} \\
+ \kappa [\omega_{kk_1k_2} (k \times k_1) (k \times k_2) + \frac{1}{2} (k \times k_1) (k_2 - k_1)] + \omega_{kk_1k_2}^2 (k_1 \times k_2) (k_1 \times k_2) + \frac{1}{2} (k \times k_1) (k_2 - k_1)] \\
+ \omega_{kk_1k_2}^2 (k_1 \times k_2) (k_1 \times k_2) - i k^2 (\omega_{kk_1k_2} - \omega_{kk_1k_2}) \\
\times (\omega_{kk_1k_2} - \omega_{kk_1k_2}) \right\} \right\}, \quad \text{(A5)}
\]

\[
N_{kk_1k_2} = \frac{4 \sqrt{2 \pi \sigma_0^2 \omega_{kk_1k_2}^2}}{G \sigma_0 \kappa^3 k_1 k_2 k^2}, \quad \text{(A6)}
\]

we can obtain

\[
V_{k_0k_0(-2k_0)} = \frac{\sigma_0 \pi G \sigma_0^2}{4 \pi N_0} \left( \pi^3 G^3 \sigma_0^3 \right)^{1/2}, \quad \text{(A7)}
\]

where

\[
c_0 = -8A_0^2 B_0^2 16 \eta^4 + 8(\gamma - 2)64 \eta^6 A_0^2 B_0 - 8Q^2 A_0^2 16 \eta^4 \quad \text{(A8)}
\]

\[
N_0 = 64 \pi^2 (\pi G \sigma_0^2 / c_s^3)^{1/2}, \quad \pi G \sigma_0^2 / c_s^3, \quad \text{(A9)}
\]

and

\[
V_{k_0k_0(-2k_0)} = \frac{\sigma_0 \pi G \sigma_0^2}{4 \pi N_0} \left( \pi^3 G^3 \sigma_0^3 \right)^{1/2}, \quad \text{(A9)}
\]

where

\[
D_0 = 8A_0^2 B_0^2 16 \eta^4 + 8(\gamma - 2)64 \eta^6 A_0^2 B_0 - 8Q^2 A_0^2 16 \eta^4 \quad \text{(A10)}
\]

Then from Eq. (2.24), omitting the unimportant terms, we have

\[
T = \frac{c_s^2}{(B_0 + 2A_0) A_0^2 B_0^2 \eta^6} \frac{G^2 \sigma_0}{c_s} \times \frac{1}{16 \times 64 \times 64} \frac{1}{c_s^2} \quad \text{(A11)}
\]

and from (2.12) we get

\[
(v'_{kk_0}) = \frac{2(Q^2 - 1) \eta / c_s}{A_0^3 k_0}. \quad \text{(A12)}
\]

Thus

\[
\beta = \frac{(2 \pi^2 T (v'_{kk_0}))}{(2 \pi^2 A_0^3 T k_0 / (Q^2 - 1) \eta / c_s)} \equiv a \left( \frac{G^2 \sigma_0 k_0}{c_s^5} \right), \quad \text{(A13)}
\]

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