Conditions that lead to a self-similar power spectrum

Hang Zhang*
Purple Mountain Observatory, Academia Sinica, Nanjing 210008, China
Xiao-qing Li
China Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing, China and Department of Physics, Nanjing Normal University, Nanjing 210097, China
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The fluid limit of the Bogolivbov-Born-Green-Kirkwood-Yvon equations admits a self-similar solution in a flat universe with an initial scale-free spectrum. By analyzing the scaling property of the Newtonian fluid equations taking advantage of the functional method, we find that the self-similar solution is required if two additional conditions are satisfied: (a) the initial fluctuation is Gaussian; (b) the decaying modes and the rotational modes in initial fluctuation have decayed before the nonlinearity comes into effect. [S0556-2821(97)05922-5]
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I. INTRODUCTION

The large scale structure of today’s Universe is believed to be the result of the gravitational amplification of small primordial fluctuation. The detected power-law shape of the correlation functions led to the suggestion that the gravitational clustering should proceed in a self-similar way. The two necessary requirements for the evolution to be self-similar are [1] (i) the universe should be spatially flat, with $\Omega_0=1$, in order to have no characteristic time or length scales in the background cosmological model; (ii) the initial power spectrum is scale-free;

$$P(k) \propto k^{n_0}$$

with $n_0$ the initial spectral index.

Although the requirements for the self-similar clustering are idealizations that cannot rigorously apply to the real universe, it is a classic problem in cosmology and has attracted considerable attention in the past. The idea of self-similarity provides a powerful aid towards a physical understanding of nonlinear gravitational clustering and in many models would expect approximate self-similar evolution over a restricted range of length and time scales [1,2].

Based on the above two assumptions, it was found by Davis and Peebles [3,4] that the fluid limit of the Bogolivbov-Born-Green-Kirkwood-Yvon (BBGKY) equations admits a self-similar solution, with the form of power spectrum

$$\langle |\delta_k|^2 \rangle = t^{3\hat{a}} F(t^{\hat{a}} k),$$

where

$$\hat{a} = \frac{4}{9 + 3n_0}.$$  

However, whether the formula (2) is a result of evolution was not confirmed by their calculations. Actually, if the initial state does not satisfy the BBGKY equations of which the time dependence has been eliminated by a self-similar assumption, this result is not guaranteed. There arises a question of what conditions are sufficient for the self-similar evolution.

Admitting a self-similar solution usually implies an internal symmetry of the evolution equations. Accordingly, in this paper we analyze the scaling property of the Newtonian evolution equations when matter dominated by using a characteristic functional and deduce the functional form of the density spectrum. Such a method has been applied to investigate the “cutoff” model where below some scale $\lambda_b$ there is no initial perturbation [5]. In the present case the initial perturbation is no longer assumed zero. It is found that the self-similar formula (2) is a result developed from linear fluctuation as long as the two additional conditions are satisfied: (a) the initial fluctuation is Gaussian; (b) the decaying terms and rotational terms in the linear approximation have been negligible before nonlinear effects become important.

As one can see, the above two additional assumptions are very popular. So it is natural to find a self-similar behavior in many models.

The remainder of the paper is organized as follows. In Sec. II we obtain the scaling property of the introduced characteristic functional. In Sec. III we derive the density perturbation spectrum. The conclusion and a discussion are given in Sec. IV.

II. SCALING PROPERTY OF A CHARACTERISTIC FUNCTIONAL

Now let us show that above two additional requirements are sufficient to deduce the self-similar evolution. After recombination, when structure formation starts, the dynamical evolution of the density perturbations with scales inside the Hubble radius is described by a pressureless and self-gravitating Newtonian fluid equation [3,6]
\[ \frac{\partial \delta}{\partial t} + \nabla \cdot u + \nabla \cdot (\delta u) = 0, \]  

(4)

\[ \frac{\partial u}{\partial t} + 2 \frac{\dot{a}}{a} u + (u, \nabla) u = -\nabla \phi, \]  

(5)

\[ \nabla^2 \phi = 4\pi G \rho a^2 \delta, \]  

(6)

where \( \nabla = \nabla_s \) is the comoving gradient, \( \delta(x) \) the contrast of the density perturbation field, \( u(x) \) the peculiar velocity, \( \phi(x) \) the potential, \( a(t) \) the cosmological scale factor, and \( \bar{\rho}(t) \) the mean density. The above fluid equations are strictly valid only on scales large compared to the strongly nonlinear clustering scale [2,7]. In the case of a flat universe, \( a \approx t^{2/3}, \bar{\rho} \approx t^{-2} \).

At early epochs, the density inhomogeneities are extremely small according to a high degree isotropy of the cosmic microwave background radiation. In this linear regime with \( \delta \ll 1 \), the nonlinear terms in Eqs. (4)–(6) can be neglected and we have the solution in Fourier components as

\[ \delta_k(t) = A_k t^{2/3} + B_k t^{-1} \]  

(7)

and

\[ u_k(t) = -\frac{2 i k}{3 \kbar} \left[ A_k t^{-1/3} - \frac{3}{2} B_k t^{-2} \right] \]
\[ + \left( \frac{t_0}{t} \right)^{4/3} \left[ u_k(t_0) + \frac{k}{\kbar} [k \cdot u_k(t_0)] \right], \]  

(8)

where \( t_0 \) is some initial epoch.

When nonlinearity becomes important, the complexity of the nonlinear dynamics makes analytical studies difficult. A perturbation theory can be applied if nonlinearity is not very strong, but it is also difficult to deal with the perturbative series. Here, the functional method [5,8] of analyzing the scale transformation property of a characteristic functional is used. The characteristic functional \( \varphi \) is introduced as

\[ \varphi[y,y_\delta] = \left\{ \exp \left[ i \int \left[ u_i(x,t) y_i(x) + \delta(x,t) y_\delta(x) \right] dx \right] \right\}, \]  

(9)

where the angle brackets denote average over an ensemble of universe with statistically equivalent perturbations and \( y_i, y_\delta \) are the functional arguments. Differentiating Eq. (9) with respect to time and using Eqs. (4)–(6), we derive a closed equation for \( \varphi \):

\[ \frac{\partial \varphi}{\partial t} = -\frac{2a}{a} \int y_i D_i \varphi dx + i \int y_i \nabla_i D_j \varphi dx \]
\[ + G \int \int y_i \nabla_i \frac{1}{|x-x'|} \bar{\rho}D_j \varphi dx dx' - \int y_\delta \nabla_i D_i \varphi dx \]
\[ + i \int y_\delta \nabla_i D_j \varphi dx, \]  

(10)

where \( D_i = \delta \delta_i / \delta x_i \) \( D_\delta = \delta \delta_\delta / \delta x \) \( D_\bar{\rho} = \delta \delta_\bar{\rho} / \delta \bar{x} \) \( D_\nabla = \delta \delta_\nabla / \delta \bar{x} \) are the variational derivatives. The Eq. (10) is complemented by initial condition. We give the initial condition at a time \( t_0 \) in the linear regime, namely, \( \varphi(t_0) \), which is fixed by \( \delta(x,t_0) \) and \( u_i(x,t_0) \) in Eq. (9).

The evolution of \( \varphi \) is completely determined by Eq. (10) and initial condition \( \varphi(t_0) \). As we can see, Eq. (10) and the initial condition are invariant with respect to the following transformation group:

\[ x = \alpha x', \quad t = \alpha^2 t', \quad y_i(x) = \alpha^{1/2} y_i'(x'), \]  

\[ y_\delta(x) = \alpha^{-1} y_\delta'(x'), \quad \delta(x,t_0) = \delta'(x',t_0'), \]  

\[ u_i(x,t_0) = \alpha^{-1/2} u_i'(x',t_0'), \]  

(11)

where \( \alpha \) and \( \beta \) are group parameters. So the final result is also invariant:

\[ \varphi(y(x), y_\delta(x), \delta(x,t_0), u_i(x,t_0)) \]
\[ = \varphi(y'(x'), y_\delta'(x'), \delta'(x',t_0'), u_i'(x',t_0')). \]  

(12)

III. DENSITY PERTURBATION SPECTRUM

In this section we will calculate the density spectrum taking advantage of the above scaling relation. To do this, we assume that a spatial ergodicity applies, i.e., the ensemble averages are equivalent to the averages taken over the configuration space. Then, the power spectrum of density field can be expressed as

\[ P(k) = \langle | \delta_k |^2 \rangle \int \langle \delta(x+r) \delta(x) \rangle e^{i k \cdot r} d^3 r \]
\[ = \int e^{i k \cdot r} \delta \frac{\delta}{\delta y_\delta(x+r)d(x+r)} \frac{\delta}{\delta y_\delta(x)dx} \varphi \bigg|_{y_\delta=y'=0} d^3 r, \]  

(13)

where \( \hat{k} \) is the unit vector of \( k \). Though we have written out explicitly the argument \( x \) in the above, the final result is not related to it. Using the scaling relation (12), we can transform Eq. (13) into a functional-dependent one on primed variables:

\[ \langle | \delta_k |^2 \rangle \approx \int e^{i k \cdot r'} \frac{\delta}{\delta y_\delta'(x'+r')d(x'+r')} \frac{\delta}{\delta y_\delta'(x')dx'} \varphi \bigg|_{y_\delta'=y'=0} \alpha^3 d^3 r'. \]  

(14)

Putting \( \alpha = k^{-1} \) and integrating over \( r' \) yields

\[ \langle | \delta_k |^2 \rangle \approx \frac{1}{v^3} F\left[ \delta'(x',t_0'), \left[ u_i'(x',t_0'), \delta'(x',t_0') \right] \right] \]  

(15)

where \( F \) is a functional of \( \delta'(x',t_0'), \delta'(x',t_0') \) and \( u_i'(x',t_0') \) of \( t' \). In isotropic case, \( F \) does not rely on direction \( \hat{k} \), as we have written. At the initial state the fluctuations is in the linear regime. If the decaying modes in Eqs. (7) and (8) are omitted, we have

\[ \delta_k(t) = A_k t^{2/3}. \]  

(16)
\[ \mathbf{u}_k(t) = -\frac{2i}{3} \frac{k}{k^2} A k t^{-1/3}. \]  

(17)

So the dominant part of the peculiar velocity field is decided completely by the density field. Therefore, if the other terms in the peculiar velocity have been decayed when nonlinearity come into effect, we can think \( F \) as a functional of \( \delta'(x', t'_0) \) only. Furthermore, the density perturbation at the initial time is assumed to be Gaussian with a scale-free spectrum

\[ P_0(k) = \langle |\delta_k(t'_0)|^2 \rangle = A k^{n_0} t'_0^{4/3}, \]

(18)

where \( A \) is a constant. If the initial spectral index \( n_0 \) is given, the statistics of the Gaussian perturbations is totally determined by the factor \( A \). So we may reasonably take \( F \) to be a function of \( A' \) instead of a functional of \( \delta'(x', t'_0) \) in Eq. (15), as an averaged result of the field \( \delta'(x', t'_0) \). Thus,

\[ F = \tilde{F}(A', t'), \]

(19)

where \( A' \) is decided by

\[ P'_0(k) = \langle |\delta'_k(t'_0)|^2 \rangle = A' k^{n_0} t'_0^{4/3}. \]

(20)

We have

\[ A' k^{n_0} t'_0^{4/3} = A k^{n_0} \alpha^{-4/3} t'_0^{4/3} \]

\[ \approx \int \langle \delta'(x' + r', t'_0) \delta'(x', t'_0) \rangle e^{ik \cdot r'} dr' \]

\[ = \int \langle \delta(x + r, t_0) \delta(x, t_0) \rangle e^{ik \cdot x} \alpha^{-3} \alpha^{-3} dr \]

\[ \approx \alpha^{-3} A(k \alpha^{-1})^{n_0} t_0^{4/3} \]

(21)

or

\[ A' k^{n_0} \alpha^{-4/3} t'_0^{4/3} = \alpha^{-3} A(k \alpha^{-1})^{n_0} t'_0^{4/3}. \]

So

\[ A' = \alpha^{-3} - n_0 + 4/3 \beta A. \]

(22)

Then from Eq. (15) we have

\[ \langle |\delta_k|^2 \rangle \propto \alpha^3 \tilde{F}(A', t') \]

\[ = \alpha^3 \tilde{F}(\alpha^{-3} - n_0 + (4/3) \beta A, \alpha^{-3} - n_0 + (4/3) \beta t') \]

\[ = \alpha^3 \tilde{F}(\alpha^{-3} - n_0 + (4/3) \beta A, \alpha^{-3} - n_0 + (4/3) \beta t') \]

\[ \propto k^{-3} \tilde{F}(A k^{3/4} n_0 t_0^{4/3}) \]

(23)

Since \( \beta \) is an arbitrary parameter, the final result will not depend on it, implying \( \delta \tilde{F}/\delta \beta = 0 \). In view of this, we have

\[ \langle |\delta_k|^2 \rangle \propto \alpha^3 \tilde{F}(\alpha^{-3} - n_0 + (4/3) \beta A, \alpha^{-3} - n_0 + (4/3) \beta t') \]

(24)

And this can be transformed to

\[ \langle |\delta_k|^2 \rangle = t^\hat{a} \tilde{F}(A t^{\hat{a} k}) \]

with

\[ \hat{a} = \frac{4}{9 + 3 n_0} \]

which is just the self-similar result (2).

IV. CONCLUSION AND DISCUSSION

In a matter-dominated flat universe with scale-free Gaussian initial spectrum, if the decaying modes and the rotational modes (also decay) in initial fluctuation are not important, then the density inhomogeneities will finally evolve self-similarly into the form (2). Though allowed by evolution equations, self-similarity may not necessarily be required. Our study gives the sufficient conditions for such an evolution. If the decaying terms are not decayed totally and still play a role in linear stage, the evolution can not be self-similar. After the decaying terms are omitted, the initial linear scale-free spectrum \( P(k) \propto k^{n_0} t^{4/3} \) is consistent with the functional form (2); but if it is not Gaussian, when nonlinearity must be considered, the functional \( F \) in Eq. (15) will still rely on the high order cumulants of the initial density. Then, the self-similar behavior may not be maintained. Our result is independent of the value of initial spectral index \( n_0 \).

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