Non-linear Stability of the Equilibrium of a System of Mass Points\textsuperscript{†} 

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Abstract By applying the non-stationary tensor virial theorem to spheroidal systems of mass points with an anisotropic velocity distribution, we have found that, when the total energy is negative, there is a unique equilibrium for a given such system. In this paper, we discuss the non-linear stability of the equilibrium. We find that, in the neighborhood of this equilibrium, spheroids execute quasi-periodic or near-quasi-periodic oscillations. Though chaoticity and orbit diffusion are present in some cases, the equilibrium is practically stable with a large stable region. This result would be useful in n-body simulations aimed for a better understanding of stellar system oscillation.

Key words: celestial mechanics—galactic equilibrium—galactic oscillation

1. INTRODUCTION

By applying the non-stationary tensor virial theorem to homogeneous spheroidal systems of mass points with locally isotropic velocity distribution, Chandrasekhar & Elbert\cite{1} and Sunder & Kochhar\cite{2} derived the following set of ordinary differential equations for the semi-axes \( a_1 (= a_2) \) and \( a_3 \)

\[
\begin{align*}
\frac{d^2 a_1^2}{dt^2} &= \frac{20E}{3M} + \frac{GM}{a_1} \left( \frac{a_1}{a_3} A_1 + 2 \frac{a_3}{a_1} A_3 \right) \\
\frac{d^2 a_3^2}{dt^2} &= \frac{20E}{3M} + \frac{GM}{a_1} \left( 4 \frac{a_1}{a_3} A_1 - \frac{a_3}{a_1} A_3 \right)
\end{align*}
\]  

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where $M$ and $E$ are the total mass and the total energy, respectively, and $A_1$ and $A_3$, the index symbols defined in the theory of the gravitational potential of ellipsoids (e.g., Binney & Tremaine). Starting from this set of differential equations, Chandrasekhar & Elbert, Sunder & Kochhar and Fu & Sun obtained the conclusion that, when $E < 0$ and the system is not far from equilibrium, the system executes finite amplitude, quasi-periodic or near-quasi-periodic oscillation.

Fu & Sun also considered spheroids with non-uniform mass distribution and non-isotropic velocity distribution. The density distribution (Sunder & Kochhar, Perek, Kerr et al., Pfenniger and Hanshima et al.) may be written as

$$\rho = \rho_c \left(1 - \sum_{i=1}^{3} \frac{x_i^2}{a_i^2}\right)^\nu.$$  

The anisotropic velocity distribution satisfies

$$K_{11} = K_{22} = \alpha K_{33} \quad (\alpha > 0),$$  

where $K_{ii}$ is the $i$th diagonal element of the kinetic-energy tensor and $\alpha$ is a positive constant. For axisymmetric spheroids near the equilibrium, (3) is a reasonable assumption. For the case of $E < 0$, the following set of ordinary differential equations was derived from the non-stationary tensor virial theorem:

$$\begin{align*}
\frac{d^2a_1}{dt^2} &= f_1(\alpha, a_1, a_3) \\
\frac{d^2a_2}{dt^2} &= f_2(\alpha, a_1, a_3),
\end{align*}$$  

where

$$\begin{align*}
f_1(\alpha, a_1, a_3) &= -\frac{6\alpha}{2\alpha + 1} + \frac{3}{(2\alpha + 1)a_1} \left[ (2\alpha - 1)\frac{a_1}{a_3} A_1 + 2\alpha \frac{a_3}{a_1} A_3 \right] \\
f_2(\alpha, a_1, a_3) &= -\frac{6}{2\alpha + 1} + \frac{3}{(2\alpha + 1)a_1} \left[ 4\frac{a_1}{a_3} A_1 - (2\alpha - 1)\frac{a_3}{a_1} A_3 \right].
\end{align*}$$  

In deriving (4), we chose the unit system such that

$$\begin{align*}
M \phi &= 1 \\
3GM^2 \psi &= 1 \\
\frac{10E}{\phi^2} &= 1,
\end{align*}$$  

where $\phi$ and $\psi$ have the same meanings as in Sunder & Kochhar. Since the unit system is independent of $\alpha$, the discussion related to time and space scaling by Sunder & Kochhar also applies to the cases of $\alpha \neq 1$.

We found that, for a given $\alpha$, there exists a unique equilibrium solution, which is linearly stable (Fu & Sun). For $\alpha > 1$, the equilibrium configuration is oblate, and the oblateness,
\[ \alpha = \frac{1}{2(1 - e_{oe}^2)} \left[ \frac{e_{oe}^3}{e_{oe} - \sqrt{1 - e_{oe}^2 \arcsin(e_{oe})}} - 1 \right], \quad (7) \]

and, for \( \alpha < 1 \), the equilibrium configuration is prolate, and the prolateness, \( e_{pe} \), satisfies

\[ \alpha = \frac{1}{2} \left[ \frac{e_{pe}^2 \ln \frac{1 + e_{pe}}{1 - e_{pe}}}{\ln \frac{1 + e_{pe}}{1 - e_{pe} - 2e_{pe}}} - 1 \right]. \quad (8) \]

The semi-axes of equilibrium, \((a_{1e}, a_{3e})\), can be explicitly expressed in terms of \( e_{oe} \) or \( e_{pe} \) in the respective cases:

\[ \begin{align*}
   a_{1e} &= a_{1oe} = \arcsin(e_{oe}) \\
   a_{3e} &= a_{3oe} = \sqrt{1 - e_{oe}^2 \arcsin(e_{oe})} / e_{oe}
\end{align*} \quad (9) \]

and

\[ \begin{align*}
   a_{1e} &= a_{1pe} = \sqrt{1 - e_{pe}^2 \ln \frac{1 + e_{pe}}{1 - e_{pe}}} / 2e_{pe} \\
   a_{3e} &= a_{3pe} = \ln \frac{1 + e_{pe}}{1 - e_{pe}} / 2e_{pe}
\end{align*} \quad (10) \]

As is well known, it is difficult to observe directly the phenomenon of oscillation of stellar systems (galaxies or their subsystems), this is because there are some intrinsic difficulties (e.g., Fu & Sun\[10\]). In fact, direct observational evidence for this phenomenon is still lacking. However, there are some indirect evidence from observation and some direct evidence from numerical experiments in favor of stellar system oscillations (e.g., Bartlett & Pike\[11\], Miller & Smith\[12\], Sellword\[13\]). As pointed out by Miller, the main difficulty in a detailed study of the oscillation by the method of n-body simulation is that little is known about the relevant robust ground state. In other words, equilibrium states with large non-linear stability regions are of particular interest. In this paper, we study the non-linear stability of the equilibrium that we previously found (Fu & Sun\[5\]). The behavior of the mass point system near the equilibrium is also discussed.

In section 2, the set of equations (4) is transformed into the standard form for a reversible system. The phase space structure and dynamical behaviors of mass point systems near the equilibrium are explored in section 3. Astronomical implications of the present work are given in section 4.

### 2. THE REVERSIBLE SYSTEM AND ITS EQUILIBRIUM SOLUTION

We know that only the region with both \( a_1 > 0 \) and \( a_3 > 0 \) is of astronomical interest. In this region, the following variable transformation is well defined (one-to-one correspondence),
real-valued and analytic:
\[
\begin{align*}
\begin{cases}
  u_1 &= a_1^2 \\
  u_3 &= a_3^2 .
\end{cases}
\end{align*}
\] (11)

Let
\[
\begin{align*}
\begin{cases}
  p_{u1} &= \dot{u}_1 \\
  p_{u3} &= \dot{u}_3 .
\end{cases}
\end{align*}
\] (12)

The system of differential equation for \( u = (u_1, u_3, p_{u1}, p_{u3}) \) may be written as
\[
\dot{u} = F(u),
\] (13)

where
\[
F(u) = (p_{u1}, p_{u3}, f_1(\alpha, \sqrt{u_1}, \sqrt{u_3}), f_3(\alpha, \sqrt{u_1}, \sqrt{u_3})).
\] (14)

Defining a matrix
\[
R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\] (15)

we have
\[
\begin{align*}
\begin{cases}
  R^2 &= \text{Identity} \\
  F(Ru) &= -RF(u) ,
\end{cases}
\end{align*}
\] (16)

that is to say, (13) is a reversible system (Moser\textsuperscript{[14]}).

Recently, several new numerical methods used in the study of Hamiltonian systems have been developed. One of them is the method of frequency map analysis (e.g., Laskar et al.\textsuperscript{[18]}). Since the phase space structure of an integrable reversible system is similar to that of an integrable Hamiltonian system (Moser\textsuperscript{[14]}), the above-mentioned methods can generally be adopted also to the study of the former. We will use the method of frequency map analysis to study the dynamical behaviors of systems of mass points near equilibrium. For this purpose, it is convenient to introduce the following variables
\[
\begin{align*}
\begin{cases}
  r_1 &= \sqrt{u_1^2 + p_{u1}^2} \\
  r_3 &= \sqrt{u_3^2 + p_{u3}^2} \\
  \theta_1 &= \arctan(p_{u1}/u_1) \pmod{2\pi} \\
  \theta_3 &= \arctan(p_{u3}/u_3) \pmod{2\pi}.
\end{cases}
\end{align*}
\] (17)

It can be easily seen that the transformation (17) is permutable with the linear reflection defined by the matrix \( R \). Therefore, the set of differential equations for \( \{r_1, r_3, \theta_1, \theta_3\} \) is also in the standard form for a reversible system (Moser\textsuperscript{[14]}). The equilibrium solution in the variables \( \{r_1, r_3, \theta_1, \theta_3\} \) may be written as
\[
\begin{align*}
\begin{cases}
  r_1 &= r_{1e} = u_{1e} = a_{1e}^2 \\
  r_3 &= r_{3e} = u_{3e} = a_{3e}^2 \\
  \theta_1 &= 0 \pmod{2\pi} \\
  \theta_3 &= 0 \pmod{2\pi}.
\end{cases}
\end{align*}
\] (18)
3. DYNAMICAL BEHAVIORS OF SYSTEMS OF MASS POINTS AROUND EQUILIBRIUM

We know that the phase space of a reversible, integrable system is composed of \( n \)-dimensional invariant tori, \( n \) being the number of degrees of freedom of the system. Motions on such tori are quasi-periodic. And, there exists a KAM-like theorem for nearly-integrable reversible systems (Moser\cite{14}). Using the technique developed by Laskar, i.e., the NAFF algorithm, we can give very good quasi-periodic approximations of quasi-periodic and weakly-chaotic solutions to the reversible system (13) in some suitable time interval \([t_0, t_0 + T]\). The approximations can be expressed as

\[
\begin{align*}
\{ u_i(t) &= Re(q_i(t_0,t_0+T)(t)) \\
p_{ui}(t) &= Im(q_i(t_0,t_0+T)(t)) \quad (i = 1, 3),
\end{align*}
\] (19)

with

\[
q_{i_0,t_0+T}(t) = a_i + \sum_{j=1}^{J} a_{ij} \exp(2\pi i \nu_{ij} t + \phi_{ij}).
\] (20)

The amplitudes \( a_{ij} \) are decreasing functions of \( j \). If a solution is quasi-periodic (ordered), then the corresponding fundamental frequencies do not change with \( t_0 \), and so all of the main frequencies \( \nu_{ij} (j = 1, \ldots, J) \) in (20) are independent of \( t_0 \). Therefore, one can distinguish chaotic and ordered solutions directly by checking whether or not \( \nu_{ij} (i = 1, 3; j = 1, \ldots, J) \) changes with \( t_0 \) (Fu et al.\cite{4}).

We have known that, when \( \alpha = 1 \), the equilibrium is practically stable (Fu et al.\cite{4}). In this paper we explore the following parameter values:

\[
\begin{align*}
\{ e_{\alpha e} &= 0.2, 0.5, 0.8 \quad \text{or} \\
\alpha &= 1.01649180, 1.12351698, 1.53001242,
\end{align*}
\] (21)

and

\[
\begin{align*}
\{ e_{\alpha e} &= 0.2, 0.5, 0.8 \quad \text{or} \\
\alpha &= 0.983831976, 0.892590497, 0.677298945.
\end{align*}
\] (22)

and, for each of the parameter values, we explore the following initial conditions:

\[
\begin{align*}
\{ r_{10} &= (1 + \frac{i}{10})r_{1e} \\
r_{30} &= (1 + \frac{j}{10})r_{3e} \\
\theta_{10} &= 0 \\
\theta_{30} &= 0,
\end{align*}
\] (23)

with

\[
\begin{align*}
\{ i &= -9, -7, \ldots, -1, +1, \ldots, +9 \\
j &= -9, -7, \ldots, -1, +1, \ldots, +9.
\end{align*}
\] (24)
Fig. 1  The distribution of the explored initial conditions (Legends: diamond-equilibriums hollow circle-ordered; cross-intermediate; filled circle-chaotic; triangle-\(a_1 \to \infty\) and \(a_2 \to 0\); upside down triangle-\(a_1 \to 0\) and \(a_2 \to \infty\))

Table 1  Orbits with frequency diffusions \(\Delta > 10^{-6}\) (Each orbit is identified by its parameter value and its initial conditions specified by (i, j).)

<table>
<thead>
<tr>
<th>No.</th>
<th>parameter</th>
<th>orbit</th>
<th>(\Delta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(e_{oe} = 0.5)</td>
<td>i=-9, j=-1</td>
<td>0.000012981</td>
</tr>
<tr>
<td>2</td>
<td>(e_{oe} = 0.8)</td>
<td>i= 7, j=-9</td>
<td>0.00000181</td>
</tr>
<tr>
<td>3</td>
<td>(e_{oe} = 0.8)</td>
<td>i= 7, j= 9</td>
<td>0.00000123</td>
</tr>
<tr>
<td>4</td>
<td>(e_{oe} = 0.8)</td>
<td>i= 9, j=-1</td>
<td>0.0000033543</td>
</tr>
<tr>
<td>5</td>
<td>(e_{oe} = 0.8)</td>
<td>i= 9, j= 1</td>
<td>0.000088</td>
</tr>
<tr>
<td>6</td>
<td>(e_{oe} = 0.8)</td>
<td>i= 9, j= 3</td>
<td>0.000012</td>
</tr>
<tr>
<td>7</td>
<td>(e_{pe} = 0.5)</td>
<td>i= 9, j= 1</td>
<td>0.0000022389</td>
</tr>
<tr>
<td>8</td>
<td>(e_{pe} = 0.5)</td>
<td>i=-9, j= 9</td>
<td>0.000004306</td>
</tr>
<tr>
<td>9</td>
<td>(e_{pe} = 0.5)</td>
<td>i=-7, j= 5</td>
<td>0.0003937855</td>
</tr>
<tr>
<td>10</td>
<td>(e_{pe} = 0.8)</td>
<td>i=-7, j=-5</td>
<td>0.00000222</td>
</tr>
</tbody>
</table>

The results are shown in Fig. 1. This figure tells us that, as the system parameter \(e_{oe}\) (or \(e_{pe}\)) increases, the number of ordered orbits decreases, which implies that more invariant tori are broken. This will result in more chaotic orbits and in greater diffusion rates of the chaotic orbits. Table 1 lists those explored orbits in which the amount of frequency diffusion from \(t = 0\) to 4096 exceeds \(10^{-6}\). As can be checked with Fig. 1, these initial conditions lie in the region of less ordered orbits. This is expected. It should be noted that even these orbits are quite regular on the time scale of the order of a Hubble time. To show this fact, we plot in Fig. 2 the chaotic orbit with the greatest \(\Delta\) (No.9 in Table 1) and show the time variation of a fundamental frequency in Fig. 3 (solid line). A quasi-periodic approximation of this solution is obtained by applying frequency analysis on the time interval \([0,4096]\). When
determined by frequency analysis, the fundamental frequency of this approximated solution varies with time. This variation, which is also shown in Fig.3 (dots), reflects the precision of determination of the fundamental frequency. As is easily seen from Fig.3, the real solution undergoes small but non-negligible frequency variations. Fast and large-scale diffusion may appear in the long run, but it is obviously unimportant on practical time scales.

The results obtained in this section can be summarized as follows. In the neighborhood of the equilibrium, spheroids execute quasi-periodic or near-quasi-periodic oscillations. Though chaoticity and orbit diffusion are present in some cases, the equilibrium is practically stable.

4. DISCUSSIONS

Recent studies show increasing evidence of stellar system oscillation (e.g., Bartlett & Pike[11], Miller & Smith[12], Sellword[13]). Though there is little direct observational confirmation of the oscillation, the indirect evidence has intrigued many researchers to study the oscillation. By successfully constructing a self-contained numerical model of an oscillating galaxy, Gerhard[18] concluded that the oscillation is theoretically possible. And, by numeri-
Fig. 3 The time variation of the smaller fundamental frequency (ν) determined by the frequency analysis. The solid line is the result for the numerically integrated solution shown in Fig. 2. The set of dots is the result for the quasi-periodic approximation of the solution.

In numerical experiments on radial virial oscillation in N-body system, David & Theuns\cite{17} obtained the conclusion that, for large $N(>500)$ the periodic oscillation predicted by Chandrasekhar & Elbert\cite{11} can survive and the period of this oscillation is in good agreement with their numerical experiments. Furthermore, some other results from numerical simulation show that oscillation is a long lasting general behavior of galaxies\cite{12}. It is generally realized that there are several physical mechanisms that act to damp global oscillations, but the above-mentioned studies seem to imply that the damping mechanisms are somewhat inefficient, at least in some cases. Another possibility is that there are some unknown stimulating mechanisms. Since even short lasting global oscillation with large amplitude can affect the stellar system evolution significantly (e.g., Lynden-Bell\cite{18}, Smith & Contopoulos\cite{19}, Fu & Sun\cite{20}), stellar system oscillation deserves further study.

While global oscillation of stellar system is evident in many N-body simulations, detailed properties are hard to extract from the numerical results. This is mainly because we do not know much about the relevant robust ground state\cite{12}.

By imposing restrictive assumptions on the mass and the velocity distributions, we present one such equilibrium in this paper. From the results obtained in the previous sections, we know that most spheroidal stellar systems, such as globular clusters and low luminosity elliptical galaxies\cite{21}, can execute quasi-periodic or near-quasi-periodic (i.e., weak-chaotic) oscillations for a long time. Generally speaking, there exist more chaotic orbits for both larger $e_{oe}$ and larger $e_{pe}$. Also, the chaoticity of the chaotic orbits generally increases with $e_{oe}$ or $e_{pe}$, such that the semi-axes diffuse more rapidly. Still, the size of the diffusion is not large, hence, not important, on the time scale of a Hubble time. Therefore, the ground state of a spheroidal stellar system found by Fu & Sun\cite{8} is practically stable.
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