CPT violating electrodynamics and Chern–Simons modified gravity

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1. Introduction

Both the Standard Model of particle physics and Einstein's general relativity are locally Lorentz and CPT invariant. Probing their violations is an important way to search for new physics and in recent years has attracted a lot of interests. Usually, studies are focused on the phenomenologies in matter and gravity sectors separately. However, there is an issue related to the consistency of the theory which needs to be examined. In a class of models of Lorentz non-covariant gravity, fixed preferred frames are introduced [1,2]. The violation effects are formulated by introducing operators like \[\mu, K^\mu\] in the Lagrangian of the matter with \[\mu\] being a fixed vector and \[K^\mu\] the matter current. The existence of the fixed preferred frames violates general coordinate covariance also. If the Lorentz invariance in the matter sector is broken in this way, it is impossible to get an energy–momentum tensor of the matter which is both symmetric and covariantly conserved. When gravity is included, this makes the Einstein's equation \[G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = -8\pi G T^{\mu\nu}\] inconsistent. Because \[G^{\mu\nu}\] is symmetric between the indices \[\mu\] and \[\nu\], and its divergence is identically zero as the result of the contracted Bianchi identity. In this Letter we provide a solution to this problem by modifying the gravity simultaneously in a non-covariant way, so that the new terms in the gravity sector balance the Lorentz violating effects in the matter sector.

As a concrete example, in this Letter we consider a matter sector where the electrodynamics is modified by a Chern–Simons term \[p_\mu A_\mu \tilde{F}^{\mu\nu}\]. This term breaks Lorentz and CPT symmetries if \[p_\mu\] is treated as an external field and the related phenomenology has been studied extensively in the literature, such as testing CPT of photons in astrophysics [3] and cosmology [4–12]. A salient feature of this modified electrodynamics is the rotations of the polarizations of propagating photons. The rotation angle \[\Delta \chi\] depends on the external field. Due to this feature, a part of E type polarization will be rotated to B type polarization for photons. This will generate TB and EB correlations in the power spectra of the cosmic microwave background radiation (CMB). Hence we can use the CMB experiments to test the Lorentz and CPT violation in this model. With homogeneous and isotropic rotation angle, in Ref. [5], two of us with Feng and Li did the simulations on the measurement of \[\Delta \chi\] with the CMBpol and PLANCK experiments. We pointed out that in such experiments the EB spectrum will be the most sensitive probe of such Lorentz and CPT violation. In [6], two of us with Feng, Xia, and Chen first found that a nonzero rotation angle \[\Delta \chi = -6.0 \pm 4.0\] deg is mildly favored by the CMB polarization data from the three-year Wilkinson Microwave Anisotropy Probe (WMAP3) observations [13–17] and the January 2003 Antarctic flight of BOOMERanG [18–20]. With the newly released five year data [9], the WMAP group gives \[\Delta \chi = -1.7 \pm 2.1\] deg, which when combined with the BOOMERanG data is [10] \[\Delta \chi = -2.6 \pm 1.9\] deg. The result by the QUAD Collaboration is \[\Delta \chi = 0.55 \pm 0.82 \pm 0.5\] deg [11] and most recently improved to \[\Delta \chi = 0.64 \pm 0.5\] deg [21].

As mentioned above, for a fixed \[p_\mu\], the electromagnetic Chern–Simons term is not invariant under the coordinate transformation. The energy–momentum tensor is the same as that of the Maxwell theory, but it is not covariantly conserved because the equation of motion is modified. This will make the Einstein's equation incon-
sistent. To preserve the consistency, we modify the Einstein’s gravity simultaneously by introducing a gravitational Chern–Simons term. As a result, the left-hand side of the gravitational equation is modified by a four-dimensional Cotton tensor which also has nonzero divergence and will match the divergence of the electromagnetic field on the right-hand side. We further in this Letter study the phenomenologies of this model on the CMB and relic gravitational waves.

Our Letter is organized as follows. In Section 2, we firstly review briefly the Maxwell–Chern–Simons theory and point out the inconsistency when including the gravity. We then introduce a gravitational Chern–Simons term and demonstrate how the theory becomes consistent; in Section 3, we study the effects of our model on CMB polarizations; in Section 4, we analyze the late-time evolution of relic gravitational waves and our result show the potential signals at high-frequency regime; Section 5 is the summary.

2. Chern–Simons modified electrodynamics and gravity

The Maxwell–Chern–Simons theory is the Maxwell electrodynamics modified by a Chern–Simons term:

$$L_F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} p_{\mu} A_\nu F^{\mu\nu} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \theta_1 F_{\mu\nu} \tilde{F}^{\mu\nu},$$

where $F_{\mu\nu}$ is the electromagnetic field tensor and $\tilde{F}^{\mu\nu} = 1/2 \times \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ is its dual, $p_\mu = V_\mu \theta_1$ characterizes the preferred frame and has dimension [E]. We use the signature (+, −, −, −) for the metric. In the second line of Eq. (1), an integration by part is used. To characterize a fixed preferred frame, $p_\mu$ is assumed to be a constant vector in spacetime, so $\theta_1 = p_\mu x^\mu + C$. Here $C$ is a constant, but it only contributes a surface term to the Lagrangian and can be set to zero. We will ignore the sources for the electromagnetic field in this Letter. When considering the minimal coupling to gravity, from the Lagrangian (1), we get the energy–momentum tensor

$$T_F^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta L}{\delta g^{\mu\nu}},$$

where $g = \det g_{\mu\nu}$. It is this energy–momentum tensor provides the source to the gravity and appears in the right-hand side of the gravitational field equation. Because the Chern–Simons term is a topological term, it does not depend on the metric and has no contribution to the energy–momentum tensor. So $T_F^{\mu\nu}$ is the same as that of the Maxwell theory,

$$T_F^{\mu\nu} = \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g^{\mu\nu} - F^{\mu\alpha} F_{\alpha\nu}.$$  

However, the equation of motion is modified as

$$\nabla_\mu F^{\mu\nu} = p_\mu \tilde{F}^{\mu\nu}. \tag{4}$$

After making use of the equation above, we find the energy–momentum tensor is not covariantly conserved,

$$\nabla_\mu T_F^{\mu\nu} = -\frac{1}{4} p^\nu F_{\mu\alpha} \tilde{F}^{\mu\alpha}. \tag{5}$$

Substitute the tensor in Eq. (3) into the Einstein equation, it becomes

$$G^{\mu\nu} = -8\pi G (T_F^{\mu\nu} + T_m^{\mu\nu}). \tag{6}$$

where $T_m^{\mu\nu}$ is the energy–momentum tensor of other matter. In this Letter we assume there are no other Lorentz violations except the Chern–Simons term in the photon sector, so $T_m^{\mu\nu}$ is symmetric and divergenceless. From Eq. (5), we see the Einstein equation is not consistent.

To solve this problem we introduce a Lorentz violating term in the gravity sector simultaneously. There are many Lorentz violating modifications of gravity discussed in the literature. In this Letter, we only consider the Chern–Simons modification proposed by Jackiw and Piu [22]. So the total Lagrangian is

$$L = \frac{1}{16\pi G} \left( R + \frac{\theta_2}{4} R \tilde{R} \right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\theta_1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} + L_m. \tag{7}$$

where $R \tilde{R} \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu} R_{\alpha\beta}$ is the gravitational Chern–Pontryagin density and $R_{\alpha\beta}$ is the Riemann tensor. Similarly, the parameter $\theta_2 = q_\mu x^\mu$, where $q_\mu$ is a constant vector with dimension $[E^{-1}]$. The variation of the action $S = \int d^4 x \sqrt{-g} L$ with respect to the metric gives the modified Einstein equation [22, 23]

$$G^{\mu\nu} + C^{\mu\nu} = -8\pi G (T_F^{\mu\nu} + T_m^{\mu\nu}), \tag{8}$$

where

$$C^{\mu\nu} = -\frac{1}{2} \left[ \nabla_\alpha \partial_\mu q_\nu \left( \epsilon^{\sigma\mu\alpha\beta} \nabla_\rho R^{\rho\nu}_{\beta} + \epsilon^{\sigma\mu\beta\alpha} \nabla_\rho R^{\rho\nu}_{\beta} \right) + \frac{1}{2} \nabla_\nu \partial_\mu q_\rho \left( \epsilon^{\sigma\rho\alpha\beta} R^{\mu\alpha}_{\beta} + \epsilon^{\sigma\mu\alpha\beta} R^{\nu\alpha}_{\beta} \right) \right]. \tag{9}$$

is the four-dimensional Cotton tensor, and its divergence gives

$$\nabla_\mu C^{\mu\nu} = \frac{1}{8} q^\nu R \tilde{R}. \tag{10}$$

So, the divergence of Eq. (8) gives the constraint

$$q^\nu R \tilde{R} = 16\pi G p^\nu F_{\mu\alpha} \tilde{F}^{\mu\alpha}. \tag{11}$$

The normal Einstein field equation is a second order partial differential equation. When modified by the gravitational Chern–Simons term, the field equation is promoted to third order due to the four-dimensional Cotton tensor shown in Eq. (9). This means the constraint (11) will not make the whole system overdetermined. But the solutions will be different from those obtained in general relativity. For example, in Ref. [22], the authors pointed out in the case $F_{\mu\alpha} \tilde{F}^{\mu\alpha} = 0$ that the Schwarzschild solution is still existent but the Kerr solution is not.

3. The effects on CMB anisotropies

In the previous section, we considered in Eq. (7) the Chern–Simons modified electromagnetic field with fixed preferred frame and introduced the gravitational Chern–Simons term to modify the gravity sector simultaneously to preserve the consistency of the theory. In this section we will study the phenomenology of this model on CMB. For keeping the rotation invariance, we assume only the temporal components of $q_\mu$ and $p_\mu$ are not vanished. During inflation the density $F_{\mu\alpha} \tilde{F}^{\mu\alpha}$ is diluted and can be set to zero. So the gravitational Chern–Simons term is constrained to be $R \tilde{R} = 0$. But at the linear order, as shown in Refs. [4, 22], the gravity is modified and the produced tensor perturbations have different intensities for different helicities. The gravitational wave has two independent polarized components denoted by $+ \times$ and $\times +$. Here it is more convenient to use the right- and left-handed circular polarized components:

$$h^R = \frac{1}{\sqrt{2}} (h^+ - i h^\times),$$

$$h^I = \frac{1}{\sqrt{2}} (h^+ + i h^\times). \tag{12}$$
The primordial gravitational waves generated during inflation freeze out after exiting the horizon. The power spectra for different handedness are defined as follows:
\[
\begin{align*}
\langle h_R^{R^*}(k_1)h_R^{R^*}(k_2) \rangle &= P_R^R \delta^3(k_1 - k_2), \\
\langle h_L^{L^*}(k_1)h_L^{L^*}(k_2) \rangle &= P_R^L \delta^3(k_1 - k_2), \\
\langle h_R^{R^*}(k_1)h_L^{L^*}(k_2) \rangle &= 0.
\end{align*}
\]
(13)

As mentioned above, due to the gravitational Chern–Simons term, the produced \( P_R^R \) and \( P_L^L \) are not equal. The discrepancy depends on \( q_0 \) and the Hubble constant \( H_{in} \) during inflation and can be denoted by the small parameter \( \epsilon = - (\pi \sqrt{2} q_0 H_{in}) [24]: \)
\[
P_R^R = \frac{1}{2} P_h(1 - \epsilon), \quad P_L^L = \frac{1}{2} P_h(1 + \epsilon), \quad P_R^L = P_h, \quad P_R^L = - \epsilon P_h.
\]
(14)

As is well known, the primordial scalar perturbations can generate the temperature and \( E \)-mode polarization perturbations in CMB. The tensor perturbations can generate \( B \)-mode perturbations besides \( T \) and \( E \). In the absence of the Chern–Simons term the generated \( B \)-modes are not correlated with \( T \) and \( E \). As mentioned above, with the gravitational Chern–Simons term, the nonvanishing \( TB \) and \( EB \) correlations would be produced at the last scattering surface, which we will explain in detail in the following.

The polarization of electromagnetic field is described by the Stokes parameters \( I, Q, U \) and \( V \). For CMB physics, the Stokes \( V \) is usually neglected because the Thomson scattering cannot produce net circular polarizations. The intensity \( I \) is invariant under coordinate transformations, but \( Q \) and \( U \) are not. The combinations \( Q \pm iU \) behave like spin-2 variables under the rotation. Given a map of temperature and polarization, we can expand the perturbations in terms of spin-weighted harmonic function as below
\[
T(\hat{n}) = \sum_{lm} a_{T,lm} Y_{lm}^*(\hat{n}),
\]
\[
( Q \pm iU)(\hat{n}) = \sum_{lm} a_{a \pm 2,lm a \pm 2} Y_{lm}^*(\hat{n}).
\]
(15)

The expressions for the expansion coefficients are
\[
a_{T,lm} = \int d\Omega \ Y_{lm}^*(\hat{n}) T(\hat{n}),
\]
\[
a_{a \pm 2,lm a \pm 2} = \int d\Omega \ Y_{lm}^*(\hat{n})(Q \pm iU)(\hat{n}).
\]
(16)

Instead of \( a_{a \pm 2,lm} \) and \( a_{-a \pm 2,lm} \), it is convenient to introduce their linear combinations
\[
a_{E,lm} = (a_{a \pm 2,lm} + a_{-a \pm 2,lm})/2,
\]
\[
a_{B,lm} = i(a_{a \pm 2,lm} - a_{-a \pm 2,lm})/2.
\]
(17)

The power spectra are defined as
\[
\tilde{C}_{l}^{XX} = 4 \pi \int k^2 dk P_h(k) \Delta_{l}^{XX}(k)^2,
\]
\[
\tilde{C}_{l}^{TE} = 4 \pi \int k^2 dk P_h(k) \Delta_{l}^{TE}(k) \Delta_{E}^{EE}(k),
\]
\[
\tilde{C}_{l}^{BB} = 4 \pi \int k^2 dk P_h(k) \Delta_{l}^{BB}(k) \Delta_{B}^{BB}(k),
\]
(18)

with the assumption of statistical isotropy, where \( X, E, B \) stand for \( T, E \) and \( B \).

In real space, using the spin raising and lowering operators \( \hat{\delta} \) and \( \hat{\delta}^* \), it useful to introduce two scalar quantities \( \hat{B}(\hat{n}) \) and \( \hat{E}(\hat{n}) \) defined as \([25]: \)
\[
\hat{B}(\hat{n}) = \frac{1}{2} \left[ \hat{\delta}^2(Q + iU) - \hat{\delta}^2(Q - iU) \right],
\]
\[
= \sum_{lm} \left[ \frac{(l+2)!}{(l-2)!} \right]^{1/2} a_{E,lm} Y_{lm}(\hat{n}).
\]
(19)

For each Fourier component, we can simply work in the coordinate frame in which \( \hat{k} \parallel \hat{z} \) and then integrate over all the Fourier modes. The generated temperature and polarization perturbations by gravitational waves can be expressed as \([25]: \)
\[
\Delta_{l}^{T}(k_0, \hat{n}, \hat{k}) = \left[ (1 - \mu^2) e^{i2\phi h_R^R(k)} + (1 - \mu^2) e^{-i2\phi h_L^L(k)} \right] \times \hat{\Delta}_{l}^{T}(k_0, \mu, k),
\]
\[
\Delta_{l}^{E}(k_0, \hat{n}, \hat{k}) = \left[ (1 + \mu^2) e^{i2\phi h_R^R(k)} + (1 + \mu^2) e^{-i2\phi h_L^L(k)} \right] \times \hat{\Delta}_{l}^{E}(k_0, \mu, k),
\]
(20)

where the superscript \( T \) denotes tensor perturbations and \( \hat{\Delta} \) are the obtained Polnarev variables \([26] \) by integrating the Boltzmann equations. We used the conformal time \( \tau \) and \( k_0 \) to denote present value, \( x = k(k_0 - \tau) \) and \( \mu = k \cdot \hat{n} \). The quantities \( \hat{\Delta}_{l}^{T}(k_0, \mu, k) \) and \( \Delta_{l}^{E}(k_0, \mu, k) \) are given by
\[
\Delta_{l}^{T}(k_0, \hat{n}, \hat{k}) = \left[ (1 + \mu^2) e^{i2\phi h_R^R(k)} + (1 + \mu^2) e^{-i2\phi h_L^L(k)} \right] \times \hat{\Delta}_{l}^{T}(k_0, \mu, k),
\]
\[
\Delta_{l}^{E}(k_0, \hat{n}, \hat{k}) = \left[ (1 + \mu^2) e^{i2\phi h_R^R(k)} + (1 + \mu^2) e^{-i2\phi h_L^L(k)} \right] \times \hat{\Delta}_{l}^{E}(k_0, \mu, k),
\]
(21)

The transfer functions \( \Delta_{l}^{T}(k_0, \mu, k) \) are defined in Eq. (30) of Ref. [25]. We can see that the effect of gravitational Chern–Simons term on CMB is only to generate the \( TB \) and \( EB \) correlations and leave other power spectra unmodified. This result is consistent with that of Ref. [27].

The scalar perturbations also have contributions to \( TT, EE \) and \( TE \). So, the total power spectra of CMB should be
\[
\tilde{C}_{l}^{TT} = C_{l}^{TT}(S) + C_{l}^{TT}(T), \quad C_{l}^{EE} = C_{l}^{EE}(S) + C_{l}^{EE}(T),
\]
\[
C_{l}^{TE} = C_{l}^{TE}(S) + C_{l}^{TE}(T), \quad C_{l}^{BB} = C_{l}^{BB}(T), \quad C_{l}^{TB} = C_{l}^{TB}(T), \quad C_{l}^{EB} = C_{l}^{EB}(T).
\]
(23)
After the perturbations of CMB generated at the last scattering surface, the polarizations of CMB photons are rotated under the influence of the electromagnetic Chern–Simons term. The rotation angle is $\Delta \chi = (p_0/2)(t_0 - t_0)$ [8], where $t_0$ is the conformal time of the last scattering. In this case, except $TT$, all other spectra are changed [6]:

$$C_{TT, \text{obs}}^{\tau} = C_{TT}^{\tau},$$
$$C_{TE, \text{obs}}^{\tau} = C_{TE}^{\tau} \cos(2\Delta \chi) - C_{TB}^{\tau} \sin(2\Delta \chi),$$
$$C_{TB, \text{obs}}^{\tau} = C_{TB}^{\tau} \sin(2\Delta \chi) + C_{TB}^{\tau} \cos(2\Delta \chi).$$

In the formulas above, the quantities with the superscript ‘obs’ are those observed after the rotation. We can see that, in the standard formal time $\tau$, form the left-hand side of Eqs. (24) are produced. Consider the modified by the Chern–Simons term.

In the energy spectrum of GWB at late-time evolution

4. Energy spectrum of GWB at late-time evolution

In this section we focus on the dynamics of gravitational waves at late-time evolution. As shown in above section, the Chern–Simons term does not contribute on the total primordial power spectrum of tensor fluctuations but only affect their propagations. Correspondingly, we expect the energy spectrum of GWB would be modified by the Chern–Simons term.

We start by giving the equation of motion for the tensor fluctuations in Fourier space [24],

$$\left(1 - \frac{\kappa^2}{\alpha^2} \right) h_k^{\text{obs}} + 2\mathcal{H} \left( 1 - \frac{1}{2} \lambda \frac{\kappa}{\alpha} \right) h_k^{\text{obs}} + \left(1 - \frac{\kappa^2}{\alpha^2} \right) k^2 h_k^{(2)} = 16\pi G \alpha^2 \sigma^2,$$

where the prime denotes the derivative with respect to the conformal time $\tau = \int_0^\tau \frac{dt}{H}$.

The subscript $s$ represents the two polarizations, with $\lambda^2 = 1$ and $\lambda^s = -1$. $\sigma^2$ is the anisotropic part of the stress tensor, constructed by the spatial components of the perturbed energy–momentum tensor. We would like to neglect it first, and then consider its contribution on transfer function later.

4.1. Canonical representation

To simplify the equation of motion for tensor fluctuations, we introduce a quantity $v_k^{(s)}$ [28], which is given by

$$v_k^{(s)}(\tau) = \sqrt{1 - \frac{\kappa^2}{\alpha^2}} \frac{k}{a} h_k^{(s)},$$

Then we redefine the variables of fluctuations as

$$\nu_k = v_k h_k^{(s)},$$

which are often viewed as generalized Mukhanov–Sasaki variables. The equation of motion for $\nu_k$ is given by

$$\nu_k^{\prime\prime} + \left( k^2 - \frac{\nu_k^{\prime\prime\prime}}{v_k^{(s)}} \right) v_k = 0.$$

Eq. (28) has an asymptotic solution when we neglect the last term $\nu_k^{\prime\prime\prime}$, which implies $|kr| \gg 1$, and it is strongly oscillating like trigonometric functions. This feature coincides with an adiabatic condition, which corresponds to the case that the effective physical wavelength is deep inside the Hubble radius. Therefore, the modes can be regarded as adiabatic when they are staying in the sub-Hubble regime with $|kr| \gg 1$, and we may impose a suitable initial condition in virtue of WKB approximation,

$$v_k^{(s)} \approx \frac{1}{\sqrt{2k}} e^{-ikr},$$

for cosmological fluctuations.

Once we have resolved the solutions to the above equations, we can obtain the tensor power spectrum $P^\nu_h$ for the polarization mode $h^s$. The GWB we may observe today should be characterized by the energy spectrum, defined by

$$\Omega_{GW}(k, \tau) = \frac{d}{d\ln k} \rho_{GW}(\tau),$$

where $\rho_{GW}(\tau)$ indicates the energy density of gravitational waves, and the parameter $\rho_{GW}(\tau)$ is the critical density of the universe. Since the GWB we observed has already re-entered the horizon, its mode should oscillate in the form of sinusoidal function. Accordingly, we can deduce the relation between the power spectrum and the energy spectrum at the scales of interests as follows

$$\Omega_{GW}(k, \tau) \approx \frac{1}{12a^3(\tau)H^3(\tau)} \sum_s P_h^{s}(k, \tau),$$

where we have used the Friedmann equation $H^3(\tau) = \frac{8\pi G}{3} \rho_c(\tau)$.

4.2. Transfer function

Now we analyze the evolution of tensor perturbations in the GWB nowadays. Since the primordial gravitational waves are distributed in every frequency, once the effective co-moving wave number is less than $aH$, the corresponding mode of gravitational waves would escape the horizon and be frozen until it re-enters the horizon. The relation between the time when tensor perturbations leave the horizon and the time when they return is $\eta_{out} H_{out} = \eta_{in} H_{in}$. Therefore, we have the conclusion that, the earlier the perturbations escape the horizon, the later they re-enter it. Moreover, once the effective co-moving wave number is larger than $aH$, the perturbations begin to oscillate like the plane wave. In the following, we will establish the relation to relate the power spectrum observed today to the primordial one. It can be described by the transfer functions which are defined as follows,

$$P_h^{s}(k, \tau) = T^s(\tau)(\tau),$$

where $\tau_i$ indicates the end of primordial inflation.

In order to make clear every possible ingredient affecting the evolvement of the GWB, it is suitable and reasonable to decompose the transfer function into three parts as follows [29],

$$T^s(\tau)(\tau) = F^s_1 F^s_2 F^s_3$$

In the above formula, $h_i^{s}(\tau)$ is the exact solution of Eq. (25); $h_i^{s}(\tau)$ is an approximate solution of Eq. (25) by neglecting the anisotropic stress tensor; and $h_i^{s}(\tau)$ is also an approximate solution which is equal to $h_i^{s}(\tau)$ if $k < aH$ while equal to plane wave if $k > aH$. First, from Eq. (29) one can see that after horizon re-entering, gravitational waves begin to oscillate with a decaying amplitude...
proportional to $1/\nu_k^4(\tau)$. Therefore, from the definition of $\tilde{h}_k^4$ we get

$$
\tilde{h}_k^4(\tau) = \begin{cases} 
\frac{A_i}{\nu_k^4(\tau)} \cos[k(\tau - \tau_k) + \phi_i], & k > aH, \\
\bar{h}_k^4(\tau), & k < aH.
\end{cases}
$$

(34)

where $\phi_i$ depends on the initial condition, $A_i^k$ is the maximum of the amplitude of oscillations, and $\tau_k$ is the conformal time when $k = aH$. Since we require this function to be continuous, there must be a matching relation that $\bar{h}_k^4(\tau_k) = [A_i^k \cos \phi_i]/\nu_k^4(\tau_k)$. Based on these relations one can get the first factor $F_1^s$ as follows

$$
F_1^s = \left(\frac{\nu_k^4(z_k)}{\nu_k^4(z)}\right)^2 \cos^2[k(\tau - \tau_k) + \phi_i]/\cos^2\phi_i.
$$

(35)

where we introduce the redshift $1 + z = a_0/a(\tau)$ in aim of showing the effects of the suppression as a result of redshift. The index "0" indicates today, and $z_k$ is the redshift when the modes re-entered the horizon $k = aH$. Note that the relation of $z_k$ and $k$ can be given by the following equation

$$
\left(\frac{k}{k_0}\right)^2 = \sum_{i} \Omega_i^{(0)} (1 + z_k) \exp \left[3 \int_0^{z_k} \frac{\Omega_i(z)}{1 + z} \right].
$$

(36)

where the sum over $i$ includes all components in the universe. Since the contributions from dark energy and the fluctuations in radiation are very small, here we ignore them and then get

$$
1 + z_k = \frac{1 + z_{eq}}{2} \left[ -1 + \sqrt{1 + \frac{4(\Omega_{m,0}^{(0)})^2}{(1 + z_{eq})^2 \Omega_m^{(0)}}} \right].
$$

(37)

where $z_{eq} = -1 + \Omega_{m,0}^{(0)}/\Omega_m^{(0)}$. The factor $F_1$ describes the redshift-suppressing effect on the primordial gravitational waves. Since this factor shows strongly oscillating behaviour which is inconspicuous to be observed in the GWB, we usually average the term $\cos^2(k(\tau - \tau_k) + \phi_i)$ and obtain $\frac{1}{4}$ for it.

Second, when considering the influence of the background equation of state of universe on the re-entry of horizon, we focus on analyzing the factor $F_2^s$. Since the background equation of state $w$ varies very slowly, it is profitable to assume that the evolution of the scale factor is of form $a = a_0(\tau/\bar{\tau})^\alpha$ with $\alpha = \frac{1}{1 + w}$. To ignore the anisotropic stress tensor $\sigma^s$ and the Chern–Simons modifications, we resolve Eq. (25) again and then have

$$
\tilde{h}_k^4(\tau) = \bar{h}_k^4(\tau) \Gamma \left(\alpha + \frac{1}{2}\right) \left(\frac{k \tau}{2}\right)^{\frac{1}{2} - \alpha} J_{\alpha - \frac{1}{2}}(k \tau),
$$

(38)

where $\Gamma$ is the Gamma function and $J_{\nu}$ is the $\nu$-th Bessel function. If $|k\tau| >> 1$, there is such a relation that

$$
\left|\frac{\tilde{u}_1(k, \tau)}{u_1(k, \tau)}\right|^2 = \frac{\Gamma^2(\alpha + \frac{1}{2})}{\pi} \left(\frac{k \tau}{2}\right)^{-2\alpha} \cos^2\left(\frac{k \tau + \alpha \pi}{2}\right).
$$

To match with Eq. (35), considering that the phase should be continuous, hence we have the solution that when tensor fluctuations re-enter the horizon the conformal time $\tau_k = -\frac{\pi}{2}$. A similar relation was obtained in [30] during primordial stage. Finally, the second factor $F_2^s$ is given by

$$
F_2^s = \frac{\Gamma^2(\alpha + \frac{1}{2})}{\pi} \left(\frac{2}{\alpha}\right)^{2\alpha} \cos^2\phi_i.
$$

(39)

The second factor shows that, when the gravitational waves re-enter the horizon, there is a "wall" lying on the horizon which affects the tensor power spectrum.

Third, during the evolution of tensor perturbations, the nonzero anisotropic stress tensor $\sigma^s$ would more or less bring some effects on the GWB. This effect is pointed out by Steven Weinberg [31], and usually the primary ingredients are the freely streaming neutrinos which damp the amplitude of the tensor power spectrum. This damping effect just makes power spectrum of tensor fluctuation times a constant but do not change the dynamics of the GWB’s evolution. A combination of analytic and numerical calculations performed in Refs. [32–36] suggests that $F_s^l = 0.80313$ for the frequency of relic gravitational waves among $10^{-16}$ Hz and $10^{-10}$ Hz is in high precision.

Eventually, we have discussed three kinds of leading corrections in the transfer function which make contributions in the evolution of the GWB. One can see that the modifications brought by the Chern–Simons term contribute mostly to the first factor $F_1$. This is because the Chern–Simons term of gravity sector mainly affect the physics at high energy scale.

4.3. Analysis of today’s GWB

Using the transfer functions, we are able to connect the primordial gravitational waves with what we observe today. To substitute Eqs. (35), (39), and the damping factor $F_1$ into (32), and making use of the primordial tensor spectrum (14) we can give today’s tensor power spectrum as follows,

$$
P_T^s(k, \tau_0) = T^s(k, \tau_0) P_T^s(k, \tau_1)
$$

$$
= \frac{1 - \lambda^4 q_0 k(1 + z_k)}{1 - \lambda^4 q_0(1 + z_k)^2} \frac{F_2^s(\alpha + \frac{1}{2})}{2\pi} \left(\frac{2}{\alpha}\right)^{2\alpha} P_T^s.
$$

(40)

When the frequency of GWB is small enough, the time for the corresponding mode re-entering the horizon is close to today, and one can see that the tensor perturbation power spectrum would agree with the standard theory very well. Therefore, we expect to find signals in the high-frequency region. To make a comparison with the normal energy spectrum of relic gravitation waves, we assume the slow-roll parameter approaches zero during inflation and the potential of the inflaton is $V_{inf} \sim M^4$ with the scale $M = 5 \times 10^{15}$ GeV. Consequently, we can obtain the semi-analytical form of the present energy spectrum of tensor perturbations with Chern–Simons modifications as follows,

$$
\Omega_{GW}(k, \tau_0) h^2 = \frac{g k^2}{(1 - \sqrt{1 + f k^2})^2} \times \sum_s \frac{1 - \lambda^4 q_0 k(1 + z_k)}{1 - \lambda^4 q_0 k},
$$

(41)

where the numerical calculations show $f = 3.10475 \times 10^{32}$, and $g$ equals to $2.15691 \times 10^{14}$ when the frequency is between $10^{-16}$ and $10^{-10}$ Hz but takes the value $3.34395 \times 10^{14}$ outside this region. In the numerical computation, we have taken $w = -1$ and $a_0 = 1$. We provide the numerical results in Fig. 1.

From Fig. 1, one can see that the energy spectra of relic gravitational waves with Chern–Simons term coincide with that in standard gravity theory at low-frequency regime, but are suppressed at high-frequency regime. Moreover, the suppressions of energy spectra strongly depend on the value of the parameter $q_0$. A similar suppression effect on energy spectrum of GWB due to a spacetime non-commutativity was found in [37], but it takes place at low-frequency regime. We can understand this effect as follows. A Lorentz-violating term often brings an effective mass for gravitons which could suppress the energy spectrum. Therefore, this suppression depends on the energy scale of the effective mass.
term. Although for both non-commutativity and Chern–Simons modifications, the Lorentz violations take place at high-energy scales, the effective mass caused by non-commutativity appears at low-frequency regime but that brought by Chern–Simons term happens at high-frequency regime.

5. Summary and conclusions

In this Letter, we pointed out that the Lorentz violations induced by fixed preferred frames in the matter sector will make the gravitational field equation inconsistent because the energy–momentum tensor cannot be both symmetric and covariantly conserved. We provided a solution to this problem by modifying the gravity simultaneously. In a concrete model, we considered the Chern–Simons modified electrodynamics which breaks Lorentz and CPT invariance. Simultaneously the gravity is also modified by a Chern–Simons term. The phenomenologies of this model on CMB and the late-time dynamics of relic gravitational waves have been studied. For these modifications, the gravitational Chern–Simons term generates non-vanished $T^B$ and $E^B$ cross-correlations of CMB at the last scattering surface and leave others unchanged. After that, the electromagnetic Chern–Simons term will rotate the generated $E$-mode polarizations to $B$-mode ones and change all but $T^T$ spectra. For the late-time evolution of the relic gravitational waves, we found that the Chern–Simons term mainly contribute to the amplitude of GWB inside the horizon. In this case, the energy spectrum of GWB is suppressed at high-frequency regime, which depends on an effective mass brought by the Chern–Simons term. From the current result, we notice that this effect is hard to be detected by recent experiments. However, the studies on gravitational models with various Lorentz violations [38,39] and their gravitational perturbations [40,41] are particularly of theoretical interests. This model deserves further studies. It will be interesting to seek for Kerr–Newmann type solutions for the spacetime metric in the region surrounding a charged and rotating celestial body.

In the literature, there is another class of Lorentz violating theories in which the Lorentz symmetry is broken spontaneously, for example, the external vector is replaced by the derivative of a dynamical scalar field, $p_\mu = V_\mu \phi$. In this case, we should include the kinetic term and the potential of $\phi$ in the action. During the evolution of the universe, the scalar field develops a nonzero $V_\mu \phi$ and the Lorentz symmetry is broken in this background. Such Lorentz violation has been used to generate the baryon number asymmetry observed in our universe [42–45]. The scalar field may be the dynamical dark energy [43] or the curvature scalar [44,45]. In this case, there is no problem of consistency and it is not necessary to modify the gravity theory [46,47]. For example, in the Maxwell–Chern–Simons theory considered in Eq. (1), if $p_\mu = V_\mu \phi$ and the kinetic term and potential of $\phi$ are included, the total energy–momentum tensor $T^{\mu\nu}_F + T^{\mu\nu}_\phi$ is symmetric and divergence free even though $T^{\mu\nu}_F$ and $T^{\mu\nu}_\phi$ are not covariantly conserved individually. The divergence of $T^{\mu\nu}_F$ is canceled by that of $T^{\mu\nu}_\phi$. An interesting effect in this case is that the rotation angle of the photon is dynamical and spacetime dependent [8].

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